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Affine $\mathrm{sl}(N)$ conformal blocks from $\mathcal{N}=2$ $\mathrm{SU}(N)$ gauge theories

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Abstract

Recently Alday and Tachikawa [1] proposed a relation between conformal blocks in a two-dimensional theory with affine sl(2) symmetry and instanton partition functions in four-dimensional conformal $\mathcal{N}=2$ SU(2) quiver gauge theories in the presence of a certain surface operator. In this paper we extend this proposal to a relation between conformal blocks in theories with affine $\mathrm{sl}(N)$ symmetry and instanton partition functions in conformal $\mathcal{N}=2$ $\mathrm{SU}(N)$ quiver gauge theories in the presence of a surface operator. We also discuss the extension to non-conformal $\mathcal{N}=2$ $\mathrm{SU}(N)$ theories.

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1 Introduction

Ever since the groundbreaking work of Seiberg and Witten [2], the study of four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry has been an important research topic. Such theories have a very rich structure and have many remarkable connections to other areas of both physics and mathematics.

Last year, building on earlier work by Witten [3], Gaiotto [4] introduced a new way of analysing $\mathcal{N} = 2$ theories by viewing them as arising from a six-dimensional theory compactified on a two-dimensional Riemann surface with punctures. In this approach one naturally expects connections between the $4d$ $\mathcal{N} = 2$ gauge theory and some conformal theory on the $2d$ Riemann surface.

The AGT relation [5] is a precise realisation of this expectation. It encompasses a relation between instanton partition functions in conformal $\mathcal{N} = 2$ theories and conformal blocks in two-dimensional conformal field theories. The original work [5] proposed a relation between conformal $4d$ SU(2) quiver gauge theories and the $2d$ Liouville theory. This relation was subsequently extended [6] to a relation between conformal $4d$ $\mathcal{N} = 2$ SU(N) theories and $2d$ A_{N-1} Toda field theories. Non-conformal $\mathcal{N} = 2$ theories have also been considered and related to two-dimensional CFT [7,8].

A natural way to extend the AGT relation is to consider the inclusion of various defects in the gauge theory. Examples include one-dimensional (line) defects (e.g. Wilson and 't Hooft loops), and three-dimensional (domain wall) defects. Such defects have been considered in [9,10], and [11], respectively.

In this paper we focus on defects which are supported on two-dimensional submanifolds, i.e. surface operators. Surface operators in $\mathcal{N} = 4$ gauge theories were extensively studied in [12] (see [13] for some similar work in $\mathcal{N} = 2$ theories). In the context of the AGT relation, surface operators have been studied in several papers [9,14–18,1].

When viewed from the six-dimensional perspective there are two ways a surface operator can arise [1]: either from a $4d$ defect wrapping the $2d$ Riemann surface, or as a $2d$ defect intersecting the $2d$ Riemann surface at a point. The second class of surface operators can be described in the dual $2d$ CFT by inserting a certain degenerate field operator localised at a point. Such surface operators were first considered in [9] and have been further studied in [14–18]. For the first class of surface operators it was recently proposed [1] that the effect of wrapping the $4d$ defect around the Riemann surface is to modify the $2d$ CFT to another $2d$ CFT. For the SU(2) quiver gauge theories it was argued that the surface operator insertion modifies the dual Liouville theory to a theory with (untwisted) affine sl(2) symmetry.

Conformal blocks in this theory should therefore be related to instanton partition functions in SU(2) quiver gauge theories in the presence of a surface operator [1] that arises from a $4d$ defect.

It was further realised in [1] that the technology to compute such instanton partition functions already exists in the mathematics literature [19–21]. Using these results several checks of the proposed relation were performed.

In this paper we extend the proposal in [1] to a relation between conformal blocks in theories with affine sl(N) symmetry and instanton partition functions in conformal $\mathcal{N} = 2$ SU(N) quiver gauge theories in the presence of a surface operator arising from

a $4d$ defect. In other words, we argue that the effect of the $4d$ defect is to replace the A_{N-1} Toda field theory and its associated $\mathcal{W}_N$-algebra symmetry by a theory with affine sl(N) symmetry. We perform several checks of the proposed relation and also extend it to non-conformal $\mathcal{N} = 2$ SU(N) theories.

In the next section we review some facts about instanton counting in SU(N) quiver gauge theories in the presence of a surface operator, and in section 3 we review the proposal in [1] and perform some additional tests using a different perturbative scheme compared to the one in [1] which allows us to sum up certain infinite sets of terms. For the rank one case we also discuss the relation to the surface operator arising from a degenerate field insertion in the Liouville theory. Then in section 4 we propose a relation between conformal blocks in a theory with affine sl(N) symmetry and instanton partition functions in SU(N) quiver gauge theories with a surface operator insertion. The extension to non-conformal theories is discussed in section 5. In the appendix some technical details are collected.

Note added: After this work was finished [22] appeared. This paper has some overlap with our results, but only considers the case of SU(2).

2 Surface operators and instanton counting

A surface operator in a four-dimensional gauge theory is a certain object supported on a two-dimensional submanifold of spacetime. One way to define a surface operator is by specifying the (singular) behaviour of the gauge field (and scalars, if present) near the submanifold where the surface operator is supported. An extensive study of surface operators in the context of the $\mathcal{N} = 4$ SU(N) gauge theories (in a flat spacetime) was carried out in [12]. There it was found that the possible types of surface operators supported on an $\mathbb{R}^2$ submanifold are in one-to-one correspondence with the so called Levi subgroups (whose classificiation in turn is in one-to-one correspondence with the various (non-trivial) ways of embedding SU(2) inside SU(N), or equivalently the number of possible ways of breaking SU(N) to a $\mathrm{U}(1)^{\ell-1}\prod_{i=1}^{\ell}\mathrm{SU}(N_i)$ (proper) subgroup). Concretely this means that for every (non-trivial) partition $N = N_1 + \ldots + N_\ell$ there is a possible surface operator. In this paper we study surface operators[1] in $4d$ SU(N) theories with $\mathcal{N} = 2$ supersymmetry; such surface operators are also classified by the Levi subgroups. For $\mathcal{N} = 2$ theories a surface operator depends on a certain number of continuous complex parameters, one for each of the abelian U(1) factors in the Levi subgroup (unbroken group)[2].

In [1] the following terminology was used: a *full* surface operator corresponds to the breaking of SU(N) to $\mathrm{U}(1)^{N-1}$ and depends on $N-1$ continuous parameters (this is the maximal number of parameters possible), whereas a *simple* surface operator corresponds to the breaking of SU(N) to SU($N-1$)$\times$U(1) and depends on one parameter.

A surface operator with a given Levi type of singularity can be realised both by $4d$ or by $2d$ defects, in the $6d$ language. In particular there will be *full* surface operators coming from $2d$ and $4d$ defects as well as *simple* surface operators coming from $2d$

[1]Throughout we assume that the surface operator is supported on an $\mathbb{R}^2$ submanifold.
[2]For $\mathcal{N} = 4$ theories the surface operators depend on four real parameters for each U(1) factor.

and $4d$ defects. Different realisations are not supposed to give rise to the same surface operator, however one may speculate that the instanton partition function may not be sensitive to the difference. We will explore this possibility in section 3.4.

In this paper surface operators that arise (in the $6d$ language) from $4d$ defects will always be full surface operators, whereas the surface operators that arise from $2d$ defects are simple surface operators. Sometimes, for convenience we will refer to the two classes just as full and simple surface operators, respectively. But the reader should keep in mind that there are in general two realisations for each Levi type of singularity.

2.1 SU(N) instanton counting in the presence of a simple surface operator

A natural question to address is how the instanton partition function in an $\mathcal{N} = 2$ gauge theory [23] (which is valid in the absence of surface operators) changes when a surface operator is present.

In [9] it was conjectured that a simple surface operator in a (mass-deformed) conformal SU(2) theory has a dual description in the Liouville theory in terms of the insertion of a certain degenerate field. It was shown that in a semi-classical limit this implies that the effect of the simple surface operator in the gauge theory can be computed from the Seiberg-Witten data, i.e. the curve and the differential. In a further development [15] it was shown how to go beyond the semi-classical analysis performed in [9] in an order-by-order ("B-model") expansion (this method also works for the cases where several simple surface operators are present).

In [15] it was also shown that by combining the conjectures in [5] and [9] (using also a result in [24]) one can obtain (conjectural) closed expressions for the gauge theory instanton partition function in SU(N) theories when simple surface operators are present (this method also works for the non-conformal cases). When lifted to $5d$ these instanton partition functions have a natural ("A-model") topological string interpretation. As emphasized by Gukov, in the topological string language a simple surface operator corresponds to a toric brane. Computing topological string partition functions with toric brane insertions leads to agreement [15, 16] with what one obtains from the combination of the conjectures in [5] and [9]. In particular, in [16] it was argued that in the topological string language this type of conjectured duality corresponds to a geometric transition (see also [18]).

For an arbitrary surface operator, generic features of the instanton expansion were discussed in [9]. For a full surface operator one can obtain exact results as we discuss next.

2.2 SU(N) instanton counting in the presence of a full surface operator

In a recent paper [1] Alday and Tachikawa proposed that the formalism needed to determine the instanton partition function in the presence of a full surface operator in an SU(N) theory has already been developed in the mathematical literature [19–21]. (Strictly speaking, it is not completely obvious that the problem solved by the mathematicians is really equivalent to the physics problem, but this is believed to be the case.)

Before we describe this construction it is convenient to first briefly recapitulate some relevant facts about the partition function, Z, in an $\mathcal{N}=2$ SU(N) quiver gauge theory (without surface operators). The partition function contains all information about the low-energy effective action and contains both perturbative (classical and one loop levels only) as well as instanton contributions; in other words

$$Z = Z_{\text{pert}}\, Z_{\text{inst}}\,. \tag{2.1}$$

The Nekrasov instanton partition function Z_{inst} is obtained from certain (regularised) integrals over the moduli space of instantons (first studied in [25]). The regularisation involves two deformation parameters, ϵ_1 and ϵ_2, that ensure that these integrals localise to isolated fixed points and can be explicitly evaluated in closed form [23]. The fixed points are labelled by a vector of Young tableaux, $\lambda = (\lambda^1, \ldots, \lambda^N)$ [23], and the resulting instanton partition function takes the form

$$Z_{\text{inst}} = \sum_{\lambda} Z_k(\lambda)\, y^k\,, \tag{2.2}$$

where the sum is over all vectors of Young tableaux λ, and the instanton number $k = |\lambda|$ is equal to the sum of the boxes in all the λ^i.

In general, a succinct way to summarise the result is in terms of a certain character. The character encodes the contribution to the instanton partition function from a given fixed point and takes the general form

$$\chi = \sum_i (\pm) e^{w_i}\,. \tag{2.3}$$

The contribution to the instanton partition function from the given fixed point (denoted $Z_k(\lambda)$ above) is given by the product over the weights w_i where the weights coming from terms in (2.3) with a minus sign contribute in the denominator and those arising from terms with a plus sign contribute in the numerator.

A basic building block is the character for a hypermultiplet of mass m transforming in the bifundamental representation of SU(N)×SU(N), which is of the general form

$$\chi_{\text{bif}}(a, \tilde{a}, \lambda, \xi, m)\,. \tag{2.4}$$

(The precise form can be found in [26], but will not be needed in this paper.) In the expression (2.4), $a = (a_1, \ldots, a_N)$ are the Coulomb moduli of the first SU(N) factor in the gauge group and $\lambda = (\lambda^1, \ldots, \lambda^N)$ is a vector of Young tableaux referring to the same SU(N) factor; $\xi = (\xi^1, \ldots, \xi^N)$ is a vector of Young tableaux referring to the second SU(N) factor and $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_N)$ are the associated Coulomb moduli. Since we want the gauge group to be SU(N) we need to impose (by hand) the restriction $\sum_i a_i = 0$ (and similarly for the $\tilde{a}_i$'s).

From the expression (2.4) one can obtain the character for other representations of interest such as the character for N hypermultiplets transforming in the fundamental representation of the first (or second) SU(N) factor, which are arise from

$$\begin{aligned}\chi_{N\,\text{funds}}(a, \lambda, \tilde{\mu}) &= \chi_{\text{bif}}(a, \tilde{\mu}, \lambda, \emptyset, 0)\,,\\ \chi_{N\,\text{funds}}(\tilde{a}, \xi, \mu) &= \chi_{\text{bif}}(\mu, \tilde{a}, \emptyset, \xi, 0)\,,\end{aligned} \tag{2.5}$$

where $\mu = (\mu_1, \ldots, \mu_N)$ and $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_N)$ denote the masses of the fundamentals *without* any restriction on $\sum_i \mu_i$ and $\sum_i \tilde{\mu}_i$, and transform under a $\mathrm{U}(N)$ flavour symmetry. (Alternatively, one can decompose μ into a part transforming under an $\mathrm{SU}(N)$ flavour symmetry plus an additional mass parameter transforming under a $\mathrm{U}(1)$ flavour symmetry.)

The character for a matter multiplet of mass m transforming in the adjoint representation of $\mathrm{SU}(N)$ is given by

$$\chi_{\mathrm{adj}}(a, \lambda, m) = \chi_{\mathrm{bif}}(a, a, \lambda, \lambda, m)\,, \tag{2.6}$$

and finally the character of the gauge vector multiplet of $\mathrm{SU}(N)$ is obtained via

$$\chi_{\mathrm{vec}}(a, \lambda) = -\chi_{\mathrm{bif}}(a, a, \lambda, \lambda, 0)\,. \tag{2.7}$$

Just as in the absence of surface operators, the instanton partition function in an $\mathrm{SU}(N)$ theory with a full surface operator involves a sum over a certain N-dimensional vector of Young tableaux $\lambda = (\lambda^1, \ldots, \lambda^N)$ where each λ^i denotes a Young tableau, or equivalently, a partition[3], i.e. $\lambda^i_1 \geq \lambda^i_2 \cdots$.

It turns out to be very convenient to view the partitions as having a periodicity, $\lambda^i \equiv \lambda^{i+N}$. Similarly, the Coulomb moduli are assumed to have the same property: $a_i \equiv a_{i+N}$. The character for a bifundamental multiplet can then be written [21, 1]

$$\begin{aligned}
\chi_{\mathrm{bif}}(a, \tilde{a}, \lambda, \xi, m) \;=\;& e^{-m} \sum_{k=1}^{N} \sum_{\ell' \geq 1} e^{a_k - \tilde{a}_{k-\ell'}} e^{\epsilon_2 (\lfloor \frac{\ell'-k}{N} \rfloor - \lfloor -\frac{k}{N} \rfloor)} \sum_{s=1}^{\xi^{k-\ell'}_{\ell'}} e^{\epsilon_1 s} \\
-\;& e^{-m} \sum_{k=1}^{N} \sum_{\ell \geq 1} \sum_{\ell' \geq 1} e^{a_{k-\ell+1} - \tilde{a}_{k-\ell'}} e^{\epsilon_2 (\lfloor \frac{\ell'-k}{N} \rfloor - \lfloor \frac{\ell-k-1}{N} \rfloor)} (e^{\epsilon_1 \xi^{k-\ell'}_{\ell'}} - 1) \sum_{s=1}^{\lambda^{k-\ell+1}_{\ell}} e^{\epsilon_1 (s - \lambda^{k-\ell+1}_{\ell})} \\
+\;& e^{-m} \sum_{k=1}^{N} \sum_{\ell \geq 1} \sum_{\ell' \geq 1} e^{a_{k-\ell+1} - \tilde{a}_{k-\ell'+1}} e^{\epsilon_2 (\lfloor \frac{\ell'-k-1}{N} \rfloor - \lfloor \frac{\ell-k-1}{N} \rfloor)} (e^{\epsilon_1 \xi^{k-\ell'+1}_{\ell'}} - 1) \sum_{s=1}^{\lambda^{k-\ell+1}_{\ell}} e^{\epsilon_1 (s - \lambda^{k-\ell+1}_{\ell})} \\
+\;& e^{-m} \sum_{k=1}^{N} \sum_{\ell \geq 1} e^{a_{k-\ell+1} - \tilde{a}_{k}} e^{\epsilon_2 (\lfloor -\frac{k}{N} \rfloor - \lfloor \frac{\ell-k-1}{N} \rfloor)} \sum_{s=1}^{\lambda^{k-\ell+1}_{\ell}} e^{\epsilon_1 (s - \lambda^{k-\ell+1}_{\ell})}
\end{aligned} \tag{2.8}$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x.

From the result (2.8) one can obtain the character for N hypermultiplets transforming in the fundamental representation of the first gauge group by setting $\xi^j = \emptyset$ for all j, cf. (2.5). Similarly, for N hypers in the fundamental representation of the second factor one sets $\lambda^i = \emptyset$, cf. (2.5). (The masses of the fundamentals are assumed to have the same periodicity as the Coulomb moduli and the partitions, i.e. $\mu_i = \mu_{i+N}$ etc.) The character for a massive matter multiplet transforming in the adjoint can also easily be obtained, cf. (2.6). Finally, the contribution from a gauge vector multiplet is obtained by setting $\xi = \lambda$ and $m = 0$, cf. (2.7).

[3] In contrast to [21] we label the components, λ^i_j, of λ^i starting from $j = 1$ rather than $j = 0$.

From these building blocks the instanton partition function for an SU(N) quiver gauge theory with bifundamental and fundamental matter multiplets in the presence of a full surface operator can be determined. For a gauge group with a single SU(N) factor the result is of the form

$$Z_{\text{inst}} = \sum_{\lambda} Z_{k_1,\ldots,k_N}(\lambda) \prod_i y_i^{k_i} \,, \tag{2.9}$$

where the instanton numbers k_i are given by [21, 1]

$$k_i = \sum_{j \geq 1} \lambda_j^{i-j+1} \,, \tag{2.10}$$

and the variables y_i (defined for $i = 1, \ldots, N$ and not assumed to be periodic in i) correspond to the $N-1$ (holomorphic) parameters of the full surface operator together with the usual instanton expansion parameter. In the general case of a quiver gauge group with several SU factors, there is a set of y_i and k_i for each factor, thus a full surface operator corresponds to breaking the complete gauge group to $\mathrm{U}(1)^r$ where r is the sum of the ranks of all factors of the quiver gauge group.

Next we consider in more detail three examples with a single SU(N) factor: the pure SU(N) theory, as well as two superconformal theories, the $\mathcal{N} = 2^*$ theory (i.e. the theory with an adjoint matter multiplet), and the theory with $N_f = 2N$ (i.e. $2N$ matter multiplets in the fundamental representation).

First we consider the terms with only one k_i non-zero. In this case, one easily sees from (2.10) that only λ^i can be non-zero and furthermore can have boxes only in the first column, i.e. only λ_1^i is $\neq 0$. This is because a non-zero λ^j with $j \neq i$ inevitably makes at least one k_j with $j \neq i$ non-zero, and the same is true for a non-zero λ_j^i with $j \geq 2$. With only λ^i non-zero and composed of only one column of height $n \equiv \lambda_1^i$, there is only one contribution at each order in the instanton expansion. From (2.8) we find that for the $\mathcal{N} = 2^*$ SU(N) theory the character corresponding to the y_i^n term in the instanton expansion becomes

$$\begin{aligned}
&(e^{-m} - 1)(e^{a_{i+1}-a_i} + 1) \sum_{s=1}^{n} e^{\epsilon_1 s} \qquad (i \leq N-1) \\
&(e^{-m} - 1)(e^{a_{i+1}-a_i+\epsilon_2} + 1) \sum_{s=1}^{n} e^{\epsilon_1 s} \qquad (i = N)
\end{aligned} \tag{2.11}$$

(for the pure SU(N) theory the result is the same but the terms involving e^{-m} are absent), whereas for the SU(N) theory with $N_f = 2N$ one finds

$$\begin{aligned}
&(-e^{a_{i+1}-a_i} + e^{\mu_{i+1}-a_i} + e^{a_i-\tilde{\mu}_i-\epsilon_1 n} - 1) \sum_{s=1}^{n} e^{\epsilon_1 s} \qquad (i \leq N-1) \\
&(-e^{a_{i+1}-a_i+\epsilon_2} + e^{\mu_{i+1}-a_i+\epsilon_2} + e^{a_i-\tilde{\mu}_i-\epsilon_1 n} - 1) \sum_{n=1}^{s} e^{\epsilon_1 s} \qquad (i = N)
\end{aligned} \tag{2.12}$$

These results lead to the following terms in the instanton partition function for the pure SU(N) theory

$$Z_{\rm inst}^{(0,i)} = \sum_{n=1}^{\infty} \frac{1}{(\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor \frac{i}{N} \rfloor + 1)_n \, n!} \left(\frac{y_i}{(\epsilon_1)^2} \right)^n . \tag{2.13}$$

Similarly, for the $\mathcal{N} = 2^*$ SU(N) theory one gets

$$Z_{\rm inst}^{(0,i)} = \sum_{n=1}^{\infty} \frac{(\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor \frac{i}{N} \rfloor + 1 - \frac{m}{\epsilon_1})_n (1 - \frac{m}{\epsilon_1})_n}{(\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor \frac{i}{N} \rfloor + 1)_n \, n!} \, (y_i)^n \, , \tag{2.14}$$

whereas for the SU(N) theory with $N_f = 2N$ the result is

$$Z_{\rm inst}^{(0,i)} = \sum_{n=1}^{\infty} \frac{(\frac{\mu_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor \frac{i}{N} \rfloor + 1)_n (\frac{\tilde{\mu}_i}{\epsilon_1} - \frac{a_i}{\epsilon_1})_n}{(\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor \frac{i}{N} \rfloor + 1)_n \, n!} \, (-y_i)^n \, . \tag{2.15}$$

In the latter two cases, $Z_{\rm inst}^{(0,i)}$ is a hypergeometric function of the form ${}_2F_1(A, B; C; y_i)$.

It is also possible to write down corrections to the above results. One natural class of corrections involve terms of the form $y_i^n \, y_j$ with $i \neq j$. Terms of this type get contributions from at most two types of Young tableaux at each order. One always gets a contribution when λ^i has only one column with n boxes and λ^j contains only one box, with all other λ^k empty. In addition, there are two special cases. First, when $j = i + 1$ one gets a contribution when λ^i has n boxes in the first column and one box in the second column with all other λ^k empty. Second, when $i = j + 1$ one gets a contribution when λ^i has $n - 1$ boxes in the first column and λ^{i-1} has one box in both the first and second columns, with all other λ^k empty. As the resulting formulæ are somewhat lengthy they have been relegated to the appendix, cf. (A.3), (A.4).

Because of the presence of $\lfloor \cdot \rfloor$ in the above formulæ, the terms involving y_N are treated differently compared to the terms involving only the other y_i. We will see in later sections that this result is reflected in the affine conformal blocks where the worldsheet coordinate z is on a different footing compared to the isospin x_i variables. The terms in the instanton partition function that are independent of y_N form an important subsector that was studied in [20]. Such terms have $k_N = 0$, which by (2.10) implies that $\lambda^j_{N-j+1} = 0$. Thus only a finite number of components of each λ^j can be non-zero. In this case the character (2.8) can be simplified. One finds after some algebra that

$$\begin{aligned} \chi_{\rm bif}(a, \tilde{a}, \lambda, \xi, m) \, |_{k_N = 0} \;&=\; e^{-m} \sum_{k=1}^{N-1} \sum_{j=1}^{k+1} \sum_{j'=1}^{k} e^{a_j - \tilde{a}_{j'}} \sum_{s=1}^{\xi^{j'}_{k-j'+1} - \lambda^j_{k-j+2}} e^{\epsilon_1 s} \\ &-\; e^{-m} \sum_{k=1}^{N-1} \sum_{j=1}^{k} \sum_{j'=1}^{k} e^{a_j - \tilde{a}_{j'}} \sum_{s=1}^{\xi^{j'}_{k-j'+1} - \lambda^j_{k-j+1}} e^{\epsilon_1 s} \, , \end{aligned} \tag{2.16}$$

which agrees with proposition 5.22 in [20] (after some changes in notation). An important thing to note is that the y_N-independent terms only depend on ϵ_1 and *not* on ϵ_2,

which is similar to the setting in [27] (see also [28, 29]). It was shown in [20] that the instanton partition function for the $\mathcal{N}=2^*$ theory with $k_N=0$ is (up to a prefactor) an eigenfunction of the quantum *trigonometric* Calogero-Sutherland model. Connections between eigenfunctions of quantum integrable systems and instanton partition functions in the presence of surface operators have also been studied in [30, 1, 17]. In particular, in [1] (see also [31]) it was argued that the instanton partition function for the $\mathcal{N}=2^*$ theory in the critical limit $\epsilon_2\to 0$ is an eigenfunction of the quantum *elliptic* Calogero-Moser model. This result is more directly related to the setup in [27].

Whereas instanton partition functions built from the character (2.8) are intimately connected with the affine sl(N) algebra the results in [20] are based on the ordinary sl(N) algebra. We will see in later sections that this fact has a natural explanation since the part of the affine conformal blocks independent of the worldsheet coordinate z is constructed from descendants that only involve the zero-modes of the affine current, which span the ordinary sl(N) Lie algebra.

It is also possible to consider quivers with more that one SU(N) factor. Here we consider one of the simplest such models, the superconformal SU(N)×SU(N) model with one matter multiplet of mass m transforming in the bifundamental representation, N multiplets with masses μ_i transforming in the fundamental representation of the first SU(N) factor and N multiplets with masses $\tilde{\mu}_i$ transforming in the fundamental representation of the second SU(N) factor.

The simplest class of terms are the ones with $k_i=n$ and $\tilde{k}_j=p$ (which arise when only $\lambda_1^i=n$ and $\xi_1^j=p$ are non-zero). For terms of this type we find that the contribution to the instanton partition function is given by

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\frac{\tilde{\mu}_i}{\epsilon_1}-\frac{a_i}{\epsilon_1})_n\,(\frac{\tilde{a}_i}{\epsilon_1}-\frac{a_i}{\epsilon_1}+\frac{m}{\epsilon_1})_n\,(\frac{\mu_{j+1}}{\epsilon_1}-\frac{\tilde{a}_j}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor+1)_p\,(\frac{a_{j+1}}{\epsilon_1}-\frac{\tilde{a}_j}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor+1-\frac{m}{\epsilon_1})_p}{(\frac{a_{i+1}}{\epsilon_1}-\frac{a_i}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor+1)_n\,n!\,(\frac{\tilde{a}_{j+1}}{\epsilon_1}-\frac{\tilde{a}_j}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor+1)_p\,p!}$$
$$\times\left[\frac{(\frac{\tilde{a}_i}{\epsilon_1}-\frac{a_i}{\epsilon_1}-p+\frac{m}{\epsilon_1})_n}{(\frac{\tilde{a}_i}{\epsilon_1}-\frac{a_i}{\epsilon_1}+\frac{m}{\epsilon_1})_n}\right]^{\delta_{ij}}\left[\frac{(\frac{\tilde{a}_j}{\epsilon_1}-\frac{a_{j+1}}{\epsilon_1}-\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor+\frac{m}{\epsilon_1})_n}{(\frac{\tilde{a}_j}{\epsilon_1}-\frac{a_{j+1}}{\epsilon_1}-\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor-p+\frac{m}{\epsilon_1})_n}\right]^{\delta_{i,j+1}}y_i^n\,\tilde{y}_j^p\,. \quad (2.17)$$

It is convenient to change notation for the masses

$$\frac{\tilde{\mu}_i}{\epsilon_1}\to\frac{\mu_{i+1}}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor+1\,,\qquad\qquad \frac{\mu_{i+1}}{\epsilon_1}\to\frac{\tilde{\mu}_i}{\epsilon_1}-\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor-1\,. \quad (2.18)$$

Using this notation the above expression becomes

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\frac{\mu_{i+1}}{\epsilon_1}-\frac{a_i}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor+1)_n\,(\frac{\tilde{a}_i}{\epsilon_1}-\frac{a_i}{\epsilon_1}+\frac{m}{\epsilon_1})_n\,(\frac{a_{j+1}}{\epsilon_1}-\frac{\tilde{a}_j}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor+1-\frac{m}{\epsilon_1})_p\,(\frac{\tilde{\mu}_j}{\epsilon_1}-\frac{\tilde{a}_j}{\epsilon_1})_p}{(\frac{a_{i+1}}{\epsilon_1}-\frac{a_i}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor+1)_n\,n!\,(\frac{\tilde{a}_{j+1}}{\epsilon_1}-\frac{\tilde{a}_j}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor+1)_p\,p!}$$
$$\times\left[\frac{(\frac{\tilde{a}_i}{\epsilon_1}-\frac{a_i}{\epsilon_1}-p+\frac{m}{\epsilon_1})_n}{(\frac{\tilde{a}_i}{\epsilon_1}-\frac{a_i}{\epsilon_1}+\frac{m}{\epsilon_1})_n}\right]^{\delta_{ij}}\left[\frac{(\frac{\tilde{a}_j}{\epsilon_1}-\frac{a_{j+1}}{\epsilon_1}-\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor+\frac{m}{\epsilon_1})_n}{(\frac{\tilde{a}_j}{\epsilon_1}-\frac{a_{j+1}}{\epsilon_1}-\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor-p+\frac{m}{\epsilon_1})_n}\right]^{\delta_{i,j+1}}y_i^n\,\tilde{y}_j^p\,. \quad (2.19)$$

In this form it is easy to see that the terms with $p=0$ or $n=0$ reduce to (2.15) with $(a_i,\mu_i,\tilde{\mu}_i)=(a_i,\mu_i,\tilde{a}_i+m)$ and $(a_i,\mu_i,\tilde{\mu}_i)=(\tilde{a}_i,a_i-m,\tilde{\mu}_i)$, respectively.

3 Affine sl(2) and surface operators in SU(2) gauge theories

In [1] it was argued that the instanton partition function in an SU(2) quiver gauge theory with a full surface operator insertion is equal to a modified version of an affine sl(2) conformal block. In this section we review and check this proposal, showing how the analytical results of the previous section can be reproduced from affine conformal blocks. We consider the four- and five-point conformal blocks on the sphere and the one-point conformal block on the torus. These are associated to the SU(2) theory with four flavours, the SU(2)×SU(2) quiver with a bifundamental hypermultiplet and two flavours in each SU(2) factor, and the $\mathcal{N}=2^*$ SU(2) gauge theory which has one adjoint hypermultiplet. In order to fix our conventions, we start by reviewing some basic facts about the affine sl(2) Lie algebra.

The commutation relations that define the untwisted affine sl(2) Lie algebra (usually denoted $\widehat{\mathrm{sl}}(2)$ or $A_1^{(1)}$) are given by

$$[J_n^0, J_m^0] = \frac{k}{2}\, n\, \delta_{n+m,0}\,, \quad [J_n^0, J_m^\pm] = \pm J_{n+m}^\pm\,, \quad [J_n^+, J_m^-] = 2J_{n+m}^0 + k\, n\, \delta_{n+m,0}\,. \tag{3.1}$$

Primary states with respect to this algebra satisfy $J_0^0|j\rangle = j|j\rangle$ and are annihilated by

$$J_{1+n}^-|j\rangle = J_{1+n}^0|j\rangle = J_n^+|j\rangle = 0 \qquad (n = 0, 1, 2, \ldots)\,, \tag{3.2}$$

which implies that

$$\langle j|J_{-1+n}^+ = \langle j|J_{-1+n}^0 = \langle j|J_n^- = 0 \qquad (n = 0, -1, -2, \ldots)\,. \tag{3.3}$$

We denote the corresponding primary field $V_j(x,z)$, where x is an isospin variable and z is the worldsheet coordinate. The action of the generators on a primary field can be expressed in terms of differential operators:

$$[J_n^A, V_j(x,z)] = z^n D^A V_j(x,z)\,, \tag{3.4}$$

where

$$D^+ = 2\, j\, x - x^2\partial_x\,, \qquad D^0 = -x\partial_x + j\,, \qquad D^- = \partial_x\,, \tag{3.5}$$

which satisfy[4]

$$[D^0, D^\pm] = \mp D^\pm\,, \qquad [D^+, D^-] = -2D^0\,. \tag{3.6}$$

The descendants of a primary state, $\langle j|$, are denoted $\langle \mathbf{n}, \mathbf{A}; j|$, where

$$\langle \mathbf{n}, \mathbf{A}; j| = \langle j|J_{n_1}^{A_1}\cdots J_{n_\ell}^{A_\ell}\,, \tag{3.7}$$

and we define the level $n = \sum_i n_i$ and charge $\Upsilon = \sum_i A_i$. For later reference, we recall that for the affine sl(2) algebra the matrix of inner products of descendants (usually called the Gram or Shapovalov matrix) satisfies

$$X_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j) = \langle \mathbf{n}, \mathbf{A}; j|\mathbf{n}', \mathbf{A}'; j\rangle \propto \delta_{n,n'}\delta_{\Upsilon,\Upsilon'}\,, \tag{3.8}$$

i.e. it is a block-diagonal matrix where each block contains only descendants with given values for the level n and charge Υ.

[4] Since $[J_n^A, [J_m^B, V_j]] = z^{n+m}D^B D^A V_j$, consistency of (3.4) implies that $[[J_n^A, J_m^B], V_j(x,z)] = -z^{n+m}[D^A, D^B]V_j(x,z)$.

3.1 Four-point conformal block on the sphere

Our first example is the four-point conformal block on the sphere. Following the proposal in [1], this should equal, up to a prefactor, the instanton partition function for the SU(2) theory with $N_f = 4$ with a full surface operator insertion. In our conventions,

$$Z_{\text{inst}} = (1-z)^{2j_2(-j_3+k/2)} \langle j_1 | V_{j_2}(1,1) \mathcal{K}(x,z) V_{j_3}(x,z) | j_4 \rangle \,, \tag{3.9}$$

where $\mathcal{K}(x,z)$ is an operator defined as

$$\mathcal{K}(x,z) = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{2n-1} \left(z^{n-1} x\, J^-_{1-n} + \frac{z^n}{x} J^+_{-n} \right) \right] . \tag{3.10}$$

The insertion of the $\mathcal{K}(x,z)$ operator is not strictly necessary for the case of the four-point block on the sphere. It is possible to reproduce the instanton partition function also without $\mathcal{K}$, by considering a small modification of the dictionary below. However, since the $\mathcal{K}$ operator is crucial when matching the higher-point conformal blocks to instanton partition functions in quiver gauge theories, we will insert a $\mathcal{K}$ operator, following the prescription in [1] (note that the expression for $\mathcal{K}$ written in [1] is equal to $\mathcal{K}(1,1)$ in our notation).

In order to reproduce the results of the previous section, we consider the following standard decomposition of the conformal block[5]

$$\begin{aligned} &\langle j_1 | V_{j_2}(1,1) \mathcal{K}(x,z) V_{j_3}(x,z) | j_4 \rangle \\ &= \sum_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'} \langle j_1 | V_{j_2}(1,1) | \mathbf{n}, \mathbf{A}; j \rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j) \langle \mathbf{n}', \mathbf{A}'; j | \mathcal{K}(x,z) V_{j_3}(x,z) | j_4 \rangle \,. \end{aligned} \tag{3.11}$$

Before we proceed with the computation of this object we would like to point out that it is also possible to reproduce the instanton partition functions (also for the quiver cases) by using a slightly different insertion, namely

$$\begin{aligned} &\langle j_1 | V_{j_2}(1,1) \mathcal{K}^\dagger(1,1) V_{j_3}(x,z) | j_4 \rangle \\ &= \sum_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'} \langle j_1 | V_{j_2}(1,1) \mathcal{K}^\dagger(1,1) | \mathbf{n}, \mathbf{A}; j \rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j) \langle \mathbf{n}', \mathbf{A}'; j | V_{j_3}(x,z) | j_4 \rangle \,. \end{aligned} \tag{3.12}$$

where

$$\mathcal{K}^\dagger(x,z) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{2n-1} \left(\frac{z^{-n+1}}{x}\, J^+_{n-1} + z^{-n} x J^-_n \right) \right] . \tag{3.13}$$

This operator will be important in section 4, but here we continue to use the expressions (3.11) and (3.10).

Note that affine and conformal invariance imply that

$$\langle j_1 | V_{j_2}(x,z) | j_3 \rangle \propto x^{j_2+j_3-j_1} z^{\Delta_1-\Delta_2-\Delta_3} \,, \tag{3.14}$$

[5] Here and in all similar expressions in the following, we omit the three-point factors. (In (3.11) the $\langle j_1 | V_{j_2}(1,1) | j \rangle \langle j | V_{j_3}(x,z) | j_4 \rangle$ factors in the denominator on the right hand side are implicit.)

where Δ_i denotes the conformal dimension of the ith state. Using this result and (3.4), it is possible to compute the conformal block (3.11) perturbatively (cf. e.g. [32]). The result is a series with only positive powers of z but both positive and negative powers of x. However, the power of x in the denominator can only be equal to or smaller than the power of z in the numerator. Two limiting cases are thus given by the z-independent terms and the subset of terms containing only powers of $\frac{z}{x}$. We start by considering the z-independent terms. These arise from descendants in the internal channel of the form $(J_0^-)^n|j\rangle$. Note that for descendants of this type, the Gram matrix is diagonal and can be trivially inverted. These terms thus lead to the following contribution

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\langle j_1|V_{j_2}(1,1)(J_0^-)^n|j\rangle\langle j|(J_0^+)^n e^{-xJ_0^-}V_{j_3}(x,z)|j_4\rangle}{\langle j|(J_0^+)^n(J_0^-)^n|j\rangle} \\
&= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\frac{(j_1-j_2-j)_n(-j-j_4+j_3)_n}{(-2j)_n}\,, \qquad (3.15)
\end{aligned}$$

where we used that $\langle j|(J_0^+)^n\mathcal{K}(x,z) = \langle j|(J_0^+)^n e^{-xJ_0^-}$ (see appendix A.3 for some additional details). In a similar way, the terms that involve only powers of $\frac{z}{x}$ can be computed. These arise from descendants in the internal channel of the form $(J_{-1}^+)^n|j\rangle$ that have a diagonal Gram matrix and lead to the contribution

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{\langle j_1|V_{j_2}(1,1)(J_{-1}^+)^n|j\rangle\langle j|(J_1^-)^n e^{-\frac{z}{x}J_{-1}^+}V_{j_3}(x,z)|j_4\rangle}{\langle j|(J_1^-)^n(J_{-1}^+)^n|j\rangle} \\
&= \sum_{n=0}^{\infty} \left(-\frac{z}{x}\right)^n\frac{1}{n!}\frac{(j-j_1-j_2)_n(j+j_4+j_3-k)_n}{(2j-k)_n}\,, \qquad (3.16)
\end{aligned}$$

where we used that $\langle j|(J_1^-)^n\mathcal{K}(x,z) = \langle j|(J_1^-)^n e^{-\frac{z}{x}J_{-1}^+}$. The expressions (3.15) and (3.16) are both hypergeometric functions of the form ${}_2F_1(A,B;C;y)$.

Next we describe the dictionary between the variables on the two sides of the conjectured equality (3.9). The worldsheet coordinate z and the isospin coordinate x are related to the instanton expansion parameters y_1 and y_2 as

$$y_1 = x\,, \qquad y_2 = \frac{z}{x}\,. \qquad (3.17)$$

Note that this identification is consistent with the fact that the instanton partition function contains only positive powers of y_1, y_2. The momenta of the external states of the conformal block are related to the hypermultiplet masses, the momentum of the internal state is related to the Coulomb modulus and the level of the affine algebra is related to the deformation parameters. The precise dictionary is

$$\begin{aligned}
j_1 &= -\frac{\epsilon_1+\epsilon_2+\mu_1-\mu_2}{2\epsilon_1}\,, \quad j_2 = -\frac{2\epsilon_1+\epsilon_2+\mu_1+\mu_2}{2\epsilon_1}\,, \quad j = -\frac{1}{2}+\frac{a_1}{\epsilon_1}\,, \\
j_3 &= -\frac{2\epsilon_1+\epsilon_2-\tilde{\mu}_1-\tilde{\mu}_2}{2\epsilon_1}\,, \quad j_4 = -\frac{\epsilon_1+\epsilon_2+\tilde{\mu}_1-\tilde{\mu}_2}{2\epsilon_1}\,, \quad k = -2-\frac{\epsilon_2}{\epsilon_1}\,. \qquad (3.18)
\end{aligned}$$

Using the dictionary (3.17), (3.18) one easily checks that (3.15), (3.16) are equal to[6] the corresponding components of the instanton partition function, (2.15).

We have also analysed the terms of the conformal block of the form $x^n z$. Such terms arise from internal states of the form

$$|1\rangle = J^+_{-1}(J^-_0)^{n+1}|j\rangle\,, \qquad |2\rangle = J^0_{-1}(J^-_0)^{n}|j\rangle\,, \qquad |3\rangle = J^-_{-1}(J^-_0)^{n-1}|j\rangle\,. \tag{3.19}$$

For any $n \geq 1$ the above states generate a 3×3 sub-block of the Gram matrix[7] and the $x^n z$ term of the conformal block is given by

$$\sum_{r,s=1}^{3} \langle j_1|V_{j_2}(1,1)|r\rangle X^{-1}_{r,s}\langle s|\mathcal{K}(x,z)V_{j_3}(x,z)|j_4\rangle\,, \tag{3.20}$$

where $X_{r,s} = \langle r|s\rangle$ with $r,s = 1,2,3$ is the relevant block of the Gram matrix. The expression (3.20) can be computed by noting that for the states considered it can be shown that

$$\langle r|\mathcal{K}(x,z) = \langle r|\mathrm{e}^{-J^-_0 x - z\left(\frac{1}{3}J^-_{-1}x + \frac{J^+_{-1}}{x}\right)} = \langle r|\mathrm{e}^{-J^-_0 x}\left[1 + z\left(-\frac{J^+_{-1}}{x} + J^0_{-1}\right)\right], \tag{3.21}$$

where we made use of the Zassenhaus formula (A.5). Using the dictionary (3.18) it can be shown that the infinite set of terms obtained from (3.20) correctly reproduce the component $Z^{(1)1,2}_{\text{inst}}$ of the SU(2) instanton partition function, (A.4). Some details of the computation can be found in appendix A.3.

3.2 Five-point conformal block on the sphere

Our second example is the five-point conformal block on the sphere. In this case we consider

$$\langle j_1|\mathcal{V}_{j_2}(1,1)\mathcal{V}_{j_3}(x,z)\mathcal{V}_{j_4}(\tilde{x},\tilde{z})|j_5\rangle\,, \tag{3.22}$$

where we introduced the notation

$$\mathcal{V}_j(x,z) = \mathcal{K}(x,z)V_j(x,z)\,. \tag{3.23}$$

In order to match the conformal block to the results for the SU(2)×SU(2) quiver gauge theory, we use the standard decomposition

$$\sum_{\mathbf{p},\mathbf{p}',\mathbf{n},\mathbf{n}'} \langle j_1|\mathcal{V}_{j_2}(1,1)|\mathbf{p};j\rangle X^{-1}_{\mathbf{p};\mathbf{p}'}(j)\langle \mathbf{p}';j|\mathcal{V}_{j_3}(x,z)|\mathbf{n};\tilde{j}\rangle X^{-1}_{\mathbf{n};\mathbf{n}'}(\tilde{j})\langle \mathbf{n}';\tilde{j}|\mathcal{V}_{j_4}(\tilde{x},\tilde{z})|j_5\rangle, \tag{3.24}$$

where for brevity we omitted the $\mathbf{A}$-type internal indices. Note that $\mathcal{V}_{j_2}$ can be replaced by V_{j_2} since $\langle j_1|\mathcal{K} = \langle j_1|$. As mentioned above it is also possible to use $\mathcal{V}_j(x,z) = V_j(x,z)\mathcal{K}^\dagger(x,z)$, but here we continue to use (3.23). Let us first focus on the terms in (3.24) with $\mathbf{n} = \mathbf{n}' = \mathbf{0}$. The non-trivial part is exactly the same four-point block

[6]Note that for these terms the prefactor in (3.9) does not give any contribution.
[7]When $n = 0$, the block reduces to a 2×2 block.

that we considered in the previous section. Summing the terms with $|\mathbf{p};j\rangle = (J_0^-)^p|j\rangle$ produces

$$\sum_{p=0}^{\infty} \frac{(j_1-j_2-j)_p(-j-\tilde{j}+j_3)_p}{(-2j)_p}\frac{(-x)^p}{p!}\,. \tag{3.25}$$

Similarly, summing the terms with $|\mathbf{p};j\rangle = (J_{-1}^+)^p|j\rangle$ gives

$$\sum_{p=0}^{\infty} \frac{(j-j_1-j_2)_p(j+\tilde{j}+j_3-k)_p}{p!(2j-k)_p}\left(-\frac{z}{x}\right)^p. \tag{3.26}$$

Next we consider the terms in (3.24) with $\mathbf{p}=\mathbf{p}'=\mathbf{0}$. The two families of internal states $|\mathbf{n};\tilde{j}\rangle = (J_0^-)^n|\tilde{j}\rangle$ and $|\mathbf{n};\tilde{j}\rangle = (J_{-1}^+)^n|\tilde{j}\rangle$ give

$$\sum_{n=0}^{\infty} \frac{(j-j_3-\tilde{j})_n(j_4-j_5-\tilde{j})_n}{n!(-2\tilde{j})_n}\left(-\frac{\tilde{x}}{x}\right)^n, \tag{3.27}$$

and

$$\sum_{n=0}^{\infty} \frac{(\tilde{j}-j-j_3)_n(\tilde{j}+j_4+j_5-k)_n}{n!(2\tilde{j}-k)_n}\left(-\frac{x\tilde{z}}{z\tilde{x}}\right)^n. \tag{3.28}$$

The above expressions (3.25)-(3.28) are all hypergeometric functions. These four hypergeometric functions can be matched to the instanton computation for the SU(2)×SU(2) quiver gauge theory (2.19). Note that in (2.19), the terms with $n=0$ sum to two hypergeometric functions in $\tilde{y}_1$ and $\tilde{y}_2$, respectively, and the $p=0$ terms sum to two hypergeometric functions in y_1 and y_2, respectively. The map between the expansion parameters is given by

$$-x=\tilde{y}_1\,,\qquad -\frac{z}{x}=\tilde{y}_2\,,\qquad -\frac{\tilde{x}}{x}=y_1\,,\qquad -\frac{x\tilde{z}}{z\tilde{x}}=y_2\,, \tag{3.29}$$

and the remaining dictionary is

$$\begin{aligned}
j_1 &= \frac{-\epsilon_1+\tilde{\mu}_1-\tilde{\mu}_2}{2\epsilon_1}\,, \qquad j_2=-\frac{\tilde{\mu}_1+\tilde{\mu}_2}{2\epsilon_1}\,, \qquad j_3=-\frac{m}{\epsilon_1}\,,\\
j_4 &= \frac{\mu_1+\mu_2}{2\epsilon_1}\,, \qquad j_5=\frac{-\epsilon_1+\mu_1-\mu_2}{2\epsilon_1}\\
j &= -\frac{1}{2}+\frac{\tilde{a}_1}{\epsilon_1}\,, \qquad \tilde{j}=-\frac{1}{2}+\frac{a_1}{\epsilon_1}\,, \qquad k=-2-\frac{\epsilon_2}{\epsilon_1}\,.
\end{aligned} \tag{3.30}$$

From the expression (3.24) it is also possible to correctly reproduce the terms in (2.19) with both $p\neq 0$ and $n\neq 0$. Using the notation

$$\begin{aligned}
|1,p;j\rangle &= (J_0^-)^p|j\rangle\,, \qquad X_{(1,p)}(j)=\langle j|(J_0^+)^p(J_0^-)^p|j\rangle\,,\\
|2,p;j\rangle &= (J_{-1}^+)^p|j\rangle\,, \qquad X_{(2,p)}(j)=\langle j|(J_1^-)^p(J_{-1}^+)^p|j\rangle\,,
\end{aligned} \tag{3.31}$$

we can summarize the result by noting that that the generic n, p term in (2.19), is equal to the term in the conformal block of the form

$$\langle j_1|\mathcal{V}_{j_2}(1,1)|r,p;j\rangle X^{-1}_{(r,p)}(j)\langle r,p;j|\mathcal{V}_{j_3}(x,z)|s,n;\tilde{j}\rangle X^{-1}_{(s,n)}(\tilde{j})\langle s,n;\tilde{j}|\mathcal{V}_{j_4}(\tilde{x},\tilde{z})|j_5\rangle, \tag{3.32}$$

where $r,s=1,2$. In conclusion, we should stress that the operator $\mathcal{K}$ was crucial for the match to the instanton result. (It is also possible to reproduce the instanton result by inserting $\mathcal{K}$ only next to V_{j_3}, rather than next to both V_{j_3} and V_{j_4} as we did above.)

3.3 One-point conformal block on the torus

Our final example is the one-point conformal block on the torus:

$$Z^{\mathcal{K}}_{\text{1pt}} = \sum_{\mathbf{n};\mathbf{A},\mathbf{n}';\mathbf{A}'} z^n x^{\Upsilon} \langle \mathbf{n},\mathbf{A};j|\mathcal{K}(x,z)V_{j_1}(x,z)|\mathbf{n}',\mathbf{A}';j\rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j)\,, \tag{3.33}$$

where $\langle \mathbf{n},\mathbf{A};j| = \langle j|J^{A_1}_{n_1}\cdots J^{A_\ell}_{n_\ell}$, $n=\sum_i n_i$ and $\Upsilon = \sum_i A_i$. It was argued in [1] that, for this case, the only effect of the insertion of the $\mathcal{K}$ operator is a prefactor:

$$Z^{\mathcal{K}}_{\text{1pt}} = (1-x-\frac{z}{x})^{-j_1} Z_{\text{1pt}}\,, \tag{3.34}$$

where Z_{1pt} is the one-point conformal block on the torus without the $\mathcal{K}$ operator, i.e.

$$Z_{\text{1pt}} = \sum_{\mathbf{n};\mathbf{A},\mathbf{n}';\mathbf{A}'} z^n x^{\Upsilon} \langle \mathbf{n},\mathbf{A};j|V_{j_1}(x,z)|\mathbf{n}',\mathbf{A}';j\rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j)\,. \tag{3.35}$$

We have checked that the relation (3.34) is satisfied for the terms that arise from internal states of the form $(J^-_0)^n|j\rangle$ or $(J^+_{-1})^n|j\rangle$. In the following we therefore focus on the conformal block without the $\mathcal{K}$ operator insertion, (3.35). As above, we first compute the z-independent terms that arise from the internal states $(J^-_0)^n|j\rangle$. These lead to the result

$$\sum_{n=0}^{\infty} x^n \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(-1)^\ell}{\ell!} \frac{(-j_1)_\ell (j_1+1)_\ell}{(-2j)_\ell}\,. \tag{3.36}$$

The terms involving powers of $\frac{z}{x}$ are due to the $(J^+_{-1})^n|j\rangle$ internal states and gives

$$\sum_{n=0}^{\infty} \left(\frac{z}{x}\right)^n \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(-1)^\ell}{\ell!} \frac{(-j_1)_\ell (j_1+1)_\ell}{(2j-k)_\ell}\,. \tag{3.37}$$

From the general result

$$(1-x)^{A-1}{}_2F_1(A,C-B;C;x) = \sum_{n=0}^{\infty} x^n \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(-1)^\ell}{\ell!} \frac{A_\ell B_\ell}{C_\ell}\,, \tag{3.38}$$

one sees that (3.36) and (3.37) can be written as

$$(1-x)^{j_1}{}_2F_1(1+j_1,-2j+j_1;-2j;x)\,, \tag{3.39}$$

and

$$\left(1-\frac{z}{x}\right)^{j_1} {}_2F_1\left(1+j_1, 2j-k+j_1; 2j-k; \frac{z}{x}\right)\,. \tag{3.40}$$

It then follows that by using (3.34) and considering the dictionary

$$x = y_1\,, \qquad \frac{z}{x} = y_2\,, \qquad j_1 = -\frac{m}{\epsilon_1}\,, \qquad j = -\frac{1}{2}+\frac{a}{\epsilon_1}\,, \qquad k = -2-\frac{\epsilon_2}{\epsilon_1}\,, \tag{3.41}$$

the conformal block precisely reproduce the result for the $\mathcal{N}=2^*$ SU(2) gauge theory (2.14) that we obtained in the previous section.

3.4 Liouville theory and surface operators in SU(2) gauge theories

In this section we explore the relation between surface operators that in the $6d$ language arise from $2d$ and $4d$ defects. For SU(2) gauge theories there is only one Levi-type of surface operator and even though the $2d$ and $4d$ defects are different objects one can investigate if the instanton partition function is sensitive to the difference.

Consider the SU(2) gauge theory with four flavours in the presence of a (simple) surface operator arising from a $2d$ defect. According to the proposal in [9], the instanton partition function should equal the Liouville conformal block with four non-degenerate primaries and one degenerate field. We have verified that, up to a prefactor, the Liouville conformal block

$$\langle\alpha_1|V_{\alpha_2}(1)V_{-\frac{b}{2}}(x)V_{\alpha_3}(z)|\alpha_4\rangle\,, \tag{3.42}$$

is indeed in agreement with the instanton computation for the SU(2) $N_f = 4$ gauge theory in the presence of a surface operator arising from a $4d$ defect that we described in section 2.[8] (For the pure SU(2) theory a similar check was performed in [17].)

As mentioned above, for the SU(2) $N_f = 4$ gauge theory, the instanton partition function can also be reproduced from an $\widehat{\mathrm{sl}}(2)$ conformal block *without* the $\mathcal{K}$ operator insertion. This implies that the result we have just described is in agreement with the Zamolodchikov-Fateev result [33], that shows that the Liouville five-point conformal block with a degenerate field insertion is equal (up to a prefactor) to the standard four-point $\widehat{\mathrm{sl}}(2)$ conformal block.

For the conformal SU(2)×SU(2) quiver gauge theory we considered above one expects a relation between the five-point $\widehat{\mathrm{sl}}(2)$ conformal block (with the $\mathcal{K}$ operator insertion) and the Liouville conformal block

$$\langle\alpha_1|V_{\alpha_2}(1)V_{-\frac{b}{2}}(x)V_{\alpha_3}(z)V_{-\frac{b}{2}}(\chi)V_{\alpha_4}(\zeta)|\alpha_5\rangle\,. \tag{3.43}$$

Using the standard decomposition, one obtains to lowest order the same structure that we found above involving four hypergeometric functions, but in this case it is not straightforward to find a relation between the two expressions (possibly one can find a map if one allows for mixing between internal/external momenta and masses/Coulomb moduli).

This indicates that already at the quiver level the instanton partition function is sensitive to the difference between $2d$ and $4d$ defects. This is perhaps not surprising since in the M-theory setup the surface operator arising from a $4d$ defect involves an M5-brane, whereas the surface operator arising from a $2d$ defect involves an M2 brane [1][9]. Also note that already for the four-point block the map is not of the form one naively would have expected since (as can be shown) $\alpha_{1,2}$ are not mapped to $j_{1,2}$. The map between full and simple surface operators deserves to be further studied.

[8] The details of this computation are collected in Appendix A.4.

[9] This is reminiscent of the situation for the $\mathcal{N} = 4$ SU(N) theories where surface operators can be constructed both using D3-branes [12] and also using D7-branes [34].

4 Affine sl(N) and surface operators in SU(N) gauge theories

In this section we make a proposal for how to extend the $\widehat{\mathrm{sl}}(2)$ analysis discussed in section 3 to $\widehat{\mathrm{sl}}(N)$. Compared to the extension of the SU(2)/Liouville results in [5] to the SU(N)/Toda results in [6], one difference is that we will be able to do computations for arbitrary N, since the affine sl(N) algebra is simpler than the $\mathcal{W}_N$ algebra.

We start by recalling some facts about the sl(N) Lie algebra. The gl(N) Lie algebra can be defined in terms of $N\times N$ matrices E^{IJ} whose only non-zero entry is a 1 at position (I,J). These matrices satisfy the commutation relations

$$[E^{IJ},E^{KL}] = \delta^{JK}E^{IL} - \delta^{LI}E^{KJ}\,. \tag{4.1}$$

For later purposes it will be convenient to use a composite index $I=(0,i)$ where $i=1,\ldots,N-1$. The generators of the sl(N) subalgebra of gl(N) are given by e.g.

$$E^i \equiv H^i = (E^{ii}-E^{i-1,i-1})/2\,, \qquad E^{i+}\equiv E^{i0}\,, \qquad E^{i-}\equiv E^{0i}\,, \qquad E^{il}\ (i\neq l)\,. \tag{4.2}$$

The commutation relations of these generators can be obtained from (4.1).

In a completely analogous convention, the generators, J^a_n, of the affine sl(N) Lie algebra (usually denoted $\widehat{\mathrm{sl}}(N)$ or $A^{(1)}_{N-1}$) are (here $i,l=1,\ldots,N-1$ and $n\in\mathbb{Z}$)

$$J^i_n\,, \qquad J^{i+}_n\,, \qquad J^{i-}_n\,, \qquad J^{il}_n \quad (i\neq l)\,. \tag{4.3}$$

Most of the commutation relations are the obvious ones induced from those of sl(N). The non-trivial ones involving the level k are[10]:

$$\begin{aligned}
[J^i_n,J^j_m] &= \frac{k}{4}\,n\,A_{ij}\,\delta_{n+m,0}\,, \qquad [J^{i+}_n,J^{i-}_m] \;=\; k\,n\,\delta_{n+m,0} + 2\sum_{s=1}^{i} J^s_{n+m}\,,\\
[J^{il}_n,J^{li}_m] &= k\,n\,\delta_{n+m,0} + \sum_{s=l+1}^{i} J^s_{n+m}\,,
\end{aligned} \tag{4.4}$$

where $A_{ij}=\langle e_i,e_j\rangle$ is the Cartan matrix. For a given value of the level k, primary states, $|j\rangle$, with respect to the $\widehat{\mathrm{sl}}(N)$ algebra are labelled by a vector j in the $N-1$ dimensional root/weight space of sl(N). Such a vector, j, can be expanded as

$$j = \sum_{i=1}^{N-1} j^i\Lambda_i\,, \tag{4.5}$$

where Λ_i are the fundamental weights of sl(N) (see appendix A.1 for a summary of our Lie algebra conventions). A primary state satisfies

$$J^i_0|j\rangle = j^i|j\rangle\,, \qquad J^{i+}_0|j\rangle = 0\,, \qquad J^{il}_0|j\rangle = 0\ \ (i>l)\,, \qquad J^A_n|j\rangle = 0\ \ (n>0)\,. \tag{4.6}$$

[10]This follows from the general result $[J^A_n,J^B_m] = f^{ABC}J^C_{n+m} + k\,m\,\delta_{m+n,0}\,\kappa^{ab}$, where κ^{ab} is the Killing form of sl(N), which in terms of the E^{IJ} $N\times N$ matrices can be written $\kappa^{IJ;KL}=\mathrm{tr}(E^{IJ}E^{KL})$.

The primary field, $V_j(x,z)$, corresponding to the primary state $|j\rangle$ satisfies

$$[J^a_n, V_j(x,z)] = z^n D^a V_j(x,z)\,, \tag{4.7}$$

where z denotes the worldsheet coordinate and x denotes a collection of isospin variables. The relation (4.7) generalises the $\widehat{\mathrm{sl}}(2)$ result (3.4). In general, the D^a's in (4.7) depend on $N(N-1)/2$ isospin variables (which equals the number of positive roots of $\mathrm{sl}(N)$); see e.g. [35] for some examples. However, as will become clear later, in the cases of main interest to us the primary fields appearing in the commutators with the J^a_n's always satisfy $j=\chi=\kappa\Lambda_{N-1}$ (or $j=\chi=\kappa\Lambda_1$). For such special primary fields we will now argue that a smaller number of isospin variables is sufficient.

There is a known realisation of $\mathrm{gl}(N)$ in terms of differential operators acting on the space spanned by $\{x_i\}$ where $i=1,\dots,N-1$ (this is the space of smallest dimension where $\mathrm{gl}(N)$ can act)[11]. More precisely, in this realisation the generators are

$$D^{00} = -2\kappa + \sum_i x_i\partial_{x_i}\,, \quad D^{i+}\equiv D^{i0} = -x_i D^{00}\,, \quad D^{i-}\equiv D^{0i} = \partial_{x_i}\,, \quad D^{il} = -x_i\partial_{x_l}\,. \tag{4.8}$$

These generators satisfy the commutation relations

$$[D^{IJ}, D^{KL}] = -\delta^{JK}D^{IL} + \delta^{LI}D^{KJ}\,, \tag{4.9}$$

i.e. the same commutation relations as the $N\times N$ matrices E^{IJ} in (4.1), but with the opposite sign on the right hand side. Note that the generators in (4.8) depend on *one* parameter, κ.

For the restriction to $\mathrm{sl}(N)$ we use the same notation as before, i.e.

$$D^i = (D^{ii}-D^{i-1,i-1})/2\,, \quad D^{i+}\equiv D^{i0}\,, \quad D^{i-}\equiv D^{0i}\,, \quad D^{il}\ \ (i\neq l)\,. \tag{4.10}$$

In terms of the D's, a highest weight representation of $\mathrm{sl}(N)$ is obtained from

$$D^i v_j(x) = j^i\, v_j(x)\,, \qquad D^{i+}v_j(x) = 0\,, \qquad D^{il}v_j(x) = 0 \quad (i>l)\,. \tag{4.11}$$

In particular, when 2κ takes the integer value n we find a finite-dimensional representation space (module) spanned by $x_1^{n_1}\cdots x_{N-1}^{n_{N-1}}$ with $0\le \sum n_i \le n = 2\kappa$. The highest weight is easily found from the above conditions: the second condition implies that $v_j(x) = x_1^{n_1}\cdots x_{N-1}^{n_{N-1}}$ with $\sum n_i = n$, and the third condition then implies that $v_j(x) = x_{N-1}^n$. Finally, we have $D^i x_{N-1}^n = -\frac{n}{2}\,\delta_{i,N-1}\,x_{N-1}^n$, i.e. the representation corresponds to the highest weight[12] $-\frac{n}{2}\Lambda_{N-1} = -\kappa\Lambda_{N-1}$. (Similarly, the lowest weight is $v_j(x)=1$, satisfying $D^i v_j = \frac{n}{2}\,\delta_{i,1}\,v_j$, corresponding to $\frac{n}{2}\Lambda_1 = \kappa\Lambda_1$.)

Taken together, these facts indicate that the $\mathrm{sl}(N)$ generators extracted from the D's in (4.8) can be used as D^a's in (4.7) when the primary field V_j has a j of the form $\kappa\Lambda_{N-1}$ ($\kappa\Lambda_1$). This proposal is very natural from the point of view of the conjecture in [1] since a full surface operator in an $\mathrm{SU}(N)$ gauge theory depends on precisely $N-1$ variables.

[11] This realisation is perhaps better known in terms of $N-1$ oscillators and dates back to [36].

[12] This is analogous to the situation in the $\mathcal{W}_N$ algebra, where a semi-degenerate state with momentum $\alpha=\kappa\Lambda_{N-1}$ becomes degenerate when $\alpha=-nb\Lambda_{N-1}$.

4.1 Four-point conformal block on the sphere

Next we turn to explicit examples and checks of the proposed relation between (slightly modified) affine sl(N) conformal blocks and instanton partition functions in SU(N) quiver gauge theories with a full surface operator insertion. Our first example is the four-point conformal block on the sphere:

$$\sum_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'} \langle j_1|\mathcal{V}_{\chi_2}(1,1)|\mathbf{n},\mathbf{A};j\rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j)\langle \mathbf{n}',\mathbf{A}';j|\mathcal{V}_{\chi_3}(x,z)|j_4\rangle\,, \tag{4.12}$$

where j, j_1, and j_4 now denote arbitrary $N-1$ dimensional vectors and $\chi_i = \kappa_i\Lambda_{N-1}$ (or $\chi_i = \kappa_i\Lambda_1$). As before, $X_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j) = \langle \mathbf{n},\mathbf{A};j|\mathbf{n}',\mathbf{A}';j\rangle$ and $|\mathbf{n}',\mathbf{A}';j\rangle$ is a descendant of the primary state $|j\rangle$:

$$|\mathbf{n},\mathbf{A};j\rangle = J^{A_1}_{-n_1}\cdots J^{A_\ell}_{-n_\ell}|j\rangle\,. \tag{4.13}$$

We conjecture that the expression (4.12) is equal (up to a prefactor) to the instanton partition function of the $\mathcal{N}=2$ SU(N) theory with $N_f = 2N$ and a full surface operator insertion. Based on the results in section 3 we expect that

$$\mathcal{V}_{\chi_i}(x,z) = V_{\chi_i}(x,z)\mathcal{K}^\dagger(x,z) \qquad \text{or} \qquad \mathcal{V}_{\chi_i}(x,z) = \mathcal{K}(x,z)V_{\chi_i}(x,z)\,. \tag{4.14}$$

We propose the following two natural generalisations of (3.10) and (3.13):

$$\begin{aligned}\mathcal{K}(x,z) \;&=\; \exp\Bigg[-\sum_{n=1}^{\infty}\frac{1}{2n-1}\Bigg(\sum_{i=1}^{N-1}\frac{z^{n-1}x^i}{i}J^{i-}_{1-n} - \sum_{i<l}\frac{z^{n-1}}{l-i}\frac{x_l}{x_i}J^{il}_{1-n}\\ &\qquad + \sum_{i=1}^{N-1}\frac{1}{i}\frac{z^n}{x_i}J^{i+}_{-n} - \sum_{i<l}\frac{z^n}{l-i}\frac{x_i}{x_l}J^{li}_{-n}\Bigg)\Bigg],\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\mathcal{K}^\dagger(x,z) \;&=\; \exp\Bigg[\sum_{n=1}^{\infty}\frac{1}{2n-1}\Bigg(\sum_{i=1}^{N-1}\frac{1}{i}\frac{z^{n-1}}{x^i}J^{i+}_{n-1} - \sum_{i<l}\frac{z^{-n+1}}{l-i}\frac{x_i}{x_l}J^{li}_{1-n}\\ &\qquad + \sum_{i=1}^{N-1}\frac{1}{i}z^{-n}x_iJ^{i-}_{n} - \sum_{i<l}\frac{z^{-n}}{l-i}\frac{x_l}{x_i}J^{il}_{-n}\Bigg)\Bigg].\end{aligned} \tag{4.16}$$

These expression are a guess based on the conjecture for $\widehat{\mathrm{sl}}(2)$ [1] together with sl(N) covariance. Note that the expressions in the exponent for different values of n commute. Also note that the zero mode part, i.e. the piece involving only J^A_0, can also be written in factorised form

$$\mathcal{K}_0 = e^{-J_0^{1-}}e^{J_0^{12}}\cdots e^{J_0^{N-2,N-1}}\,, \qquad\qquad \mathcal{K}_0^\dagger = e^{J_0^{1+}}e^{-J_0^{21}}\cdots e^{-J_0^{N-1,N-2}}\,. \tag{4.17}$$

We should stress that in our explicit examples below we will only check parts of these expressions. It would be desirable to have further checks and a better understanding of $\mathcal{K}^\dagger$ and $\mathcal{K}$.

Note that affine and conformal invariance implies that

$$\langle j_1|V_{\chi_2}(x,z)|j_3\rangle \propto z^{\Delta_1-\Delta_2-\Delta_3}\prod_{i=1}^{N-1} x_i^{2(\frac{\kappa_2}{N}-\langle h_{i+1},j_3\rangle+\langle h_{i+1},j_1\rangle)}. \tag{4.18}$$

Here h_i ($i=1,\ldots,N$) are the weights of the fundamental representation (see appendix A.1 for a summary of our Lie algebra conventions). This result can be derived by inserting J_0^i into the three-point function and using

$$\begin{aligned} j_1^i\langle j_1|V_{\chi_2}(x,z)|j_3\rangle &= \langle j_1|J_0^iV_{\chi_2}(x,z)|j_3\rangle = \langle j_1|([J_0^i,V_{\chi_2}(x,z)]+V_{\chi_2}(x,z)J_0^i)|j_3\rangle \\ &= (\tfrac{1}{2}[D^{ii}-D^{i-1,i-1}]+j_3^i)\langle j_1|V_{\chi_2}(x,z)|j_3\rangle\,. \end{aligned} \tag{4.19}$$

(The result of this argument shows that the exponents of the x_i can be written in terms of j_r^i where $r=1,2,3$, i.e in terms of the components of $j_{1,2,3}$ in the expansion (4.5); using the conventions in appendix A.1 the exponents can then be written in the above form.) Note that (4.18) reduces to (3.14) when $N=2$.

Using (4.7) and (4.18) the four-point conformal block can be computed perturbatively. The result is a series with only positive powers of z but both positive and negative powers of the x_i.

As a first example we consider the z-independent terms. Among all possible such terms, there are $N-1$ subsets, each involving a power series in one particular combination of the x_i's, that only receive contributions from one type of descendants. More precisely, these subsets involve only terms of the form x_1^n or only $(\frac{x_i}{x_{i-1}})^n$ for a fixed i with $i=2,\ldots,N-1$, and arise from descendants $\langle\mathbf{n}',\mathbf{A}';j|$ involving only J_0^{1+} or only $J_0^{i,i-1}$ for a fixed i with $i=2,\ldots,N-1$.

The only non-zero $X_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j)$ involving the relevant descendants are

$$\langle j|(J_0^{1+})^n(J_0^{1-})^n|j\rangle = n!(-2j^1)_n(-1)^n\,,\quad \langle j|(J_0^{i,i-1})^n(J_0^{i-1,i})^n|j\rangle = n!(-2j^i)_n(-1)^n, \tag{4.20}$$

where for the second expression $i=2\ldots,N-1$; note that the two results fit together nicely in one sl(N) covariant expression.

Let us first focus on the x_1^n terms and compute the four-point block *without* the insertion of $\mathcal{K}$ or $\mathcal{K}^\dagger$, i.e. we use $\mathcal{V}_{\chi_i}=V_{\chi_i}$. In this case we find

$$\begin{aligned} &\frac{\langle j_1|V_{\chi_2}(1,1)(J_0^{1-})^n|j\rangle\langle j|(J_0^{1+})^nV_{\chi_3}(x,z)|j_4\rangle}{\langle j|(J_0^{1+})^n(J_0^{1-})^n|j\rangle} \\ =\;& \frac{(-2\frac{\kappa_2}{N}-2\langle h_2,j_1\rangle+2\langle h_2,j\rangle)_n(-2\frac{\kappa_3}{N}+2\langle h_1,j_4\rangle-2\langle h_1,j\rangle)_n}{n!(-2j^1)_n}(x_1)^n\,. \end{aligned} \tag{4.21}$$

Summing the x_1^n terms then leads to the hypergeometric function ${}_2F_1(A,B;C;x)$ with

$$A=-2\frac{\kappa_2}{N}+2\langle h_2,j\rangle-2\langle h_2,j_1\rangle\,,\quad B=-2\frac{\kappa_3}{N}+2\langle h_1,j_4\rangle-2\langle h_1,j\rangle\,,\quad C=-2j^1\,. \tag{4.22}$$

Similarly if we use $\mathcal{V}_{\chi_i}(x,z)=\mathcal{K}(x,z)V_{\chi_i}(x,z)$ instead, we find

$$\begin{aligned} &\frac{\langle j_1|V_{\chi_2}(1,1)(J_0^{1-})^n|j\rangle\langle j|(J_0^{1+})^ne^{-x_1J_0^{1-}}V_{\chi_3}(x,z)|j_4\rangle}{\langle j|(J_0^{1+})^n(J_0^{1-})^n|j\rangle} \\ =\;& \frac{(-\frac{2\kappa_2}{N}-2\langle h_2,j_1\rangle+2\langle h_2,j\rangle)_n(\frac{2\kappa_3}{N}-2\langle h_1,j_4\rangle+2\langle h_2,j\rangle)_n}{n!(-2j^1)_n}(-x_1)^n\,. \end{aligned} \tag{4.23}$$

Note that the full $\mathcal{K}$ (4.15) is not needed here; only a part contributes. Finally if we use $\mathcal{V}_{\chi_i}(x,z) = V_{\chi_i}(x,z)\mathcal{K}^\dagger(x,z)$ we find

$$\begin{aligned}
&\frac{\langle j_1|V_{\chi_2}(1,1)e^{J_0^{1+}}(J_0^{1-})^n|j\rangle\langle j|(J_0^{1+})^n V_{\chi_3}(x,z)|j_4\rangle}{\langle j|(J_0^{1+})^n(J_0^{1-})^n|j\rangle} \\
&= \frac{(\frac{2\kappa_2}{N}+2\langle h_2,j_1\rangle-2\langle h_1,j\rangle)_n(-\frac{2\kappa_3}{N}+2\langle h_1,j_4\rangle-2\langle h_1,j\rangle)_n}{n!(-2j^1)_n}(-x_1)^n\,, \quad (4.24)
\end{aligned}$$

where again the full $\mathcal{K}^\dagger$ is not needed. The above expressions should be compared to the y_1^n terms in the instanton partition function (2.15), which is also of hypergeometric form. By equating the denominators we see that (up to a constant) we should equate $j^1 \propto a_1 - a_2$. This result in turn implies that $\langle h_i, j\rangle \propto a_i$ (for $i = 1,2$ and again up to a constant). Since the y_1^n terms in (2.15) only involve a_1 in the numerator, it seems that only (4.24) can equal the instanton result. For this reason we will use insertions of $\mathcal{K}^\dagger$ in the remainder of this section.

For the $\widehat{\mathrm{sl}}(2)$ four-point conformal block all three of the above possibilities could be matched to the instanton result (using minor modifications in the dictionary). It is easy to see why: for sl(2) we have that $h_1 = -h_2$. For the SU(2) quiver theories, as pointed out in section 3, insertions of either $\mathcal{K}$ or $\mathcal{K}^\dagger$ are needed. It is conceivable that also in higher rank theories one can use $\mathcal{K}$ insertions provided that one uses suitable conventions, but here we will use $\mathcal{K}^\dagger$ since it results in expressions that can be matched to the instanton results in a straightforward and natural way.

Before we proceed with the computation, let us mention another property of the insertions of $\mathcal{K}$ and $\mathcal{K}^\dagger$ that may turn out to be important. As can be seen from the above expressions (4.21), (4.23) the effect of the insertion of $\mathcal{K}$ is to turn ${}_2F_1(A,B;C;x_1)$ into ${}_2F_1(A,C-B;C;-x_1)$. Similarly, as can be seen from (4.24) the $\mathcal{K}^\dagger$ insertion results in ${}_2F_1(C-A,B;C;-x_1)$. Hypergeometric functions satisfy various identities, such as

$$ {}_2F_1(A,C-B;C;x) = (1-x)^{-A}{}_2F_1(A,B;C;-\frac{x}{1-x})\,, \qquad (4.25)$$

therefore a possible alternative to the insertion of $\mathcal{K}^\dagger$ ($\mathcal{K}$) might be to change variables instead (or to pick a different solution of the hypergeometric differential equation), but we will not pursue this idea here.

Returning to the computation we find that in addition to (4.24) we also have

$$\begin{aligned}
&\frac{\langle j_1|V_{\chi_2}(1,1)e^{-J_0^{i,i-1}}(J_0^{i-1,i})^n|j\rangle\langle j|(J_0^{i,i-1})^n V_{\chi_3}(x,z)|j_4\rangle}{\langle j|(J_0^{i,i-1})^n(J_0^{i-1,i})^n|j\rangle} \\
&= \frac{(\frac{2\kappa_2}{N}+2\langle h_{i+1},j_1\rangle-2\langle h_i,j\rangle)_n(-\frac{2\kappa_3}{N}+2\langle h_i,j_4\rangle-2\langle h_i,j\rangle)_n}{n!(-2j^i)_n}\left(-\frac{x_i}{x_{i-1}}\right)^n, \quad (4.26)
\end{aligned}$$

where $i = 2,\ldots,N-1$. Let us emphasize again that in the above expressions the full $\mathcal{K}^\dagger$ (3.13) is not needed. It is easy to see that all J_n^A dependence drops out for $n > 0$. To see that also most of the J_0^A dependence drops out requires a bit more thought.

For the z-dependent part, terms of the form $(\frac{z}{x_{N-1}})^n$ also only receive contributions from one source, namely from the the descendants involving only $J_1^{N-1,-}$. The relevant

component of the Gram matrix $X_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j)$ is

$$\langle j|(J_1^{N-1,-})^n(J_{-1}^{N-1,+})^n|j\rangle = n!(2\,[\sum_{i=1}^{N-1} j^i] - k)_n(-1)^n\,. \tag{4.27}$$

Using this result we find:

$$\begin{aligned}&\frac{\langle j_1|V_{\chi_2}(1,1)e^{J_1^{N-1,-}}(J_{-1}^{N-1,+})^n|j\rangle\langle j|(J_1^{N-1,-})^nV_{\chi_3}(x,z)|j_4\rangle}{\langle j|(J_1^{N-1,-})^n(J_{-1}^{N-1+})^n|j\rangle}\\ =\;&\frac{(\frac{2\kappa_2}{N}+2\langle h_1,j_1\rangle-2\langle h_N,j\rangle-k)_n(-\frac{2\kappa_3}{N}+2\langle h_N,j_4\rangle-2\langle h_N,j\rangle)_n}{n!(2\,[\sum_{i=1}^{N-1} j^i]-k)_n}\left(-\frac{z}{x_{N-1}}\right)^n.\end{aligned} \tag{4.28}$$

Note that $j^i = \langle e_i, j\rangle = \langle u_i - u_{i+1}, j\rangle$. This implies that $\sum_{i=1}^{N-1} j^i = -\langle u_N - u_1, j\rangle$ which shows that (4.28) fits together nicely with the other results (4.24), (4.26) provided that one identifies $u_{i+N} = u_i$.

The above results should be compared with the instanton result (2.15). We propose that

$$y_1 = x_1\,, \qquad y_{i+1} = \frac{x_{i+1}}{x_i} \quad (1 \leq i \leq N-2)\,, \qquad y_N = \frac{z}{x_{N-1}}\,. \tag{4.29}$$

Non-trivial evidence for this identification comes from the fact that, as is easy to see, all terms in the expansion of the conformal block can be written as a power series with only positive powers of the y_i's (the instanton result (2.15) is also a power series in N variables y_i with only positive powers). To match the denominators of (4.24), (4.26) to (2.15) we identify

$$j^i = -\frac{1}{2} + \frac{a_i - a_{i+1}}{2\epsilon_1}\,, \qquad\qquad k = -N - \frac{\epsilon_2}{\epsilon_1}\,. \tag{4.30}$$

Given that $\sum_{i=1}^N a_i = 0$, it follows from the above formula that

$$2\,[\sum_{i=1}^{N-1} j^i] - k = \frac{a_1 - a_N}{\epsilon_1} + 1 + \frac{\epsilon_2}{\epsilon_1}\,, \tag{4.31}$$

and thus the denominator in (4.28) also agrees with the instanton result. Defining $a = \sum_i^N a_i u_i$, the relation (4.30) can also be written in various other ways

$$\langle e_i, j\rangle = \frac{1}{2}\langle e_i, \frac{a}{\epsilon_1} - \rho\rangle\,, \quad \langle u_i, j\rangle = \frac{1}{2}\langle u_i, \frac{a}{\epsilon_1} - \rho\rangle\,, \quad \langle h_i, j\rangle = \frac{1}{2}\langle h_i, \frac{a}{\epsilon_1} - \rho\rangle\,. \tag{4.32}$$

We also propose the following dictionary for the masses

$$\frac{\tilde{\mu}_i}{2\epsilon_1} = -\frac{\kappa_3}{N} + \langle h_i, j_4 + \frac{\rho}{2}\rangle\,, \qquad \frac{\mu_i}{2\epsilon_1} = \frac{\kappa_2}{N} + \langle h_i, j_1 + \frac{\rho}{2}\rangle\,, \tag{4.33}$$

which leads to complete agreement between (2.15) and the results in this section. (Note that the $\widehat{\mathrm{sl}}(2)$ version of the map (4.33) is slightly simpler than the one used in (3.18) which arose from an expression where $\mathcal{K}$ was used instead of $\mathcal{K}^\dagger$.)

One can also compute corrections to the above expressions. One particular class of such corrections involve terms of the form $y_i^n\, y_l$ with $l \neq i$. The first thing to note is that when $l \neq i \pm 1$ there is only one possible way to obtain such terms. This result agrees with the structure of the instanton expansion (A.4). When i, l belong to the range $2, \ldots, N-1$ the contributing descendants are of the form

$$\langle j|(J_0^{i,i-1})^n (J_0^{l,l-1})\,. \tag{4.34}$$

The corresponding terms in the conformal block are easily computed. When $l \neq i \pm 1$, one finds (4.26) multiplied by

$$\frac{(\frac{2\kappa_2}{N} + 2\langle h_{l+1}, j_1\rangle - 2\langle h_l, j\rangle)(-\frac{2\kappa_3}{N} + 2\langle h_l, j_4\rangle - 2\langle h_l, j\rangle)}{(-2j^l)} \left(-\frac{x_l}{x_{l-1}}\right). \tag{4.35}$$

This result is easily seen to agree with the instanton result (A.4) (when $l \neq i \pm 1$) using the maps (4.29), (4.30), and (4.33). When $l = i \pm 1$ the situation is slightly more involved. We have checked that the $x_1^n x_2$ terms correctly reproduce the instanton result (A.4). This computation is sensitive to other terms in $\mathcal{K}^\dagger$ besides the ones appearing in the zeroth order analysis; some formulæ are collected in appendix A.3.

4.2 Five-point conformal block on the sphere

Our next example is the five-point conformal block (for brevity we suppressed the $\mathbf{A}, \mathbf{A}'$-type labels):

$$\sum_{\mathbf{n},\mathbf{n}',\mathbf{m},\mathbf{m}'} \langle j_1|\mathcal{V}_{\chi_2}(1,1)|\mathbf{n}; j\rangle X^{-1}_{\mathbf{n};\mathbf{n}'}(j)\langle \mathbf{n}'; j|\mathcal{V}_{\chi_3}(x,z)|\mathbf{m}; \tilde{\jmath}\rangle X^{-1}_{\mathbf{m};\mathbf{m}'}(\tilde{\jmath})\langle \mathbf{m}'; \tilde{\jmath}|\mathcal{V}_{\chi_4}(\tilde{x}, \tilde{z})|j_5\rangle\,, \tag{4.36}$$

where $\mathcal{V}_{\chi_i}(x,z) = V_{\chi_i}(x,z)\mathcal{K}^\dagger(x,z)$ and we have inserted two complete sets of states $|\mathbf{n}; j\rangle$ and $|\mathbf{m}; \tilde{\jmath}\rangle$. Using (4.7) the conformal block can be computed perturbatively in powers of x_i, z and $\tilde{x}_i, \tilde{z}$. Just like in the $\widehat{\mathrm{sl}}(2)$ analysis, certain terms with $\mathbf{m} = \mathbf{m}' = \mathbf{0}$ or $\mathbf{n} = \mathbf{n}' = \mathbf{0}$ can easily be computed. The terms with $\mathbf{m} = \mathbf{m}' = \mathbf{0}$ lead to hypergeometric functions of the type determined in the four-point analysis above:

$$\begin{aligned}
&\frac{\langle j_1|V_{\chi_2}(1,1)e^{J_0^{1+}}(J_0^{1-})^n|j\rangle\langle j|(J_0^{1+})^n V_{\chi_3}(x,z)|\tilde{\jmath}\rangle}{\langle j|(J_0^{1+})^n (J_0^{1-})^n|j\rangle} \\
&= \frac{(\frac{2\kappa_2}{N} + 2\langle h_2, j_1\rangle - 2\langle h_1, j\rangle)_n(-\frac{2\kappa_3}{N} + 2\langle h_1, \tilde{\jmath}\rangle - 2\langle h_1, j\rangle)_n}{n!(-2j^1)_n}(-x_1)^n\,, \\
&\frac{\langle j_1|V_{\chi_2}(1,1)e^{-J_0^{i,i-1}}(J_0^{i-1,i})^n|j\rangle\langle j|(J_0^{i,i-1})^n V_{\chi_3}(x,z)|\tilde{\jmath}\rangle}{\langle j|(J_0^{i,i-1})^n (J_0^{i-1,i})^n|j\rangle} \\
&= \frac{(\frac{2\kappa_2}{N} + 2\langle h_{i+1}, j_1\rangle - 2\langle h_i, j\rangle)_n(-\frac{2\kappa_3}{N} + 2\langle h_i, \tilde{\jmath}\rangle - 2\langle h_i, j\rangle)_n}{n!(-2j^i)_n}\left(-\frac{x_i}{x_{i-1}}\right)^n,
\end{aligned} \tag{4.37}$$

and

$$\frac{\langle j_1|V_{\chi_2}(1,1)e^{J_1^{N-1,-}}(J_{-1}^{N-1,+})^n|j\rangle\langle j|(J_1^{N-1,-})^n V_{\chi_3}(x,z)|\tilde{\jmath}\rangle}{\langle j|(J_1^{N-1,-})^n(J_{-1}^{N-1+})^n|j\rangle} \tag{4.38}$$

$$= \frac{(\frac{2\kappa_2}{N}+2\langle h_1,j_1\rangle-2\langle h_N,j\rangle-k)_n(-\frac{2\kappa_3}{N}+2\langle h_N,\tilde{\jmath}\rangle-2\langle h_N,j\rangle)_n}{n!(2\,[\sum_{i=1}^{N-1}j^i]-k)_n}\left(-\frac{z}{x_{N-1}}\right)^n. \tag{4.39}$$

Similarly, when $\mathbf{n}=\mathbf{n}'=\mathbf{0}$ we obtain hypergeometric functions from

$$\frac{\langle j|V_{\chi_3}(x,z)e^{\frac{1}{x_1}J_0^{1+}}(J_0^{1-})^n|\tilde{\jmath}\rangle\langle\tilde{\jmath}|(J_0^{1+})^n V_{\chi_4}(\tilde{x},\tilde{z})|j_5\rangle}{\langle\tilde{\jmath}|(J_0^{1+})^n(J_0^{1-})^n|\tilde{\jmath}\rangle}$$

$$= \frac{(\frac{2\kappa_3}{N}+2\langle h_2,j\rangle-2\langle h_1,\tilde{\jmath}\rangle)_n(-\frac{2\kappa_4}{N}+2\langle h_1,j_5\rangle-2\langle h_1,\tilde{\jmath}\rangle)_n}{n!(-2\tilde{\jmath}^1)_n}\left(-\frac{\tilde{x}_1}{x_1}\right)^n,$$

$$\frac{\langle j|V_{\chi_3}(x,z)e^{-\frac{x_{i-1}}{x_i}J_0^{i,i-1}}(J_0^{i-1,i})^n|\tilde{\jmath}\rangle\langle\tilde{\jmath}|(J_0^{i,i-1})^n V_{\chi_4}(\tilde{x},\tilde{z})|j_5\rangle}{\langle j|(J_0^{i,i-1})^n(J_0^{i-1,i})^n|j\rangle}$$

$$= \frac{(\frac{2\kappa_3}{N}+2\langle h_{i+1},j\rangle-2\langle h_i,\tilde{\jmath}\rangle)_n(-\frac{2\kappa_4}{N}+2\langle h_i,j_5\rangle-2\langle h_i,\tilde{\jmath}\rangle)_n}{n!(-2\tilde{\jmath}^i)_n}\left(-\frac{\tilde{x}_i\,x_{i-1}}{\tilde{x}_{i-1}\,x_i}\right)^n, \tag{4.40}$$

and

$$\frac{\langle j|V_{\chi_3}(x,z)e^{\frac{x_{N-1}}{z}J_1^{N-1,-}}(J_{-1}^{N-1,+})^n|\tilde{\jmath}\rangle\langle\tilde{\jmath}|(J_1^{N-1,-})^n V_{\chi_4}(\tilde{x},\tilde{z})|j_5\rangle}{\langle j|(J_1^{N-1,-})^n(J_{-1}^{N-1+})^n|j\rangle}$$

$$= \frac{(\frac{2\kappa_3}{N}+2\langle h_1,j\rangle-2\langle h_N,\tilde{\jmath}\rangle-k)_n(-\frac{2\kappa_4}{N}+2\langle h_N,j_5\rangle-2\langle h_N,\tilde{\jmath}\rangle)_n}{n!(2\,[\sum_{i=1}^{N-1}\tilde{\jmath}^i]-k)_n}\left(-\frac{\tilde{z}\,x_{N-1}}{z\,\tilde{x}_{N-1}}\right)^n. \tag{4.41}$$

The precise dictionary which equates the above expressions to the instanton result (2.19) is

$$\begin{aligned} y_1 &= -x_1\,, \qquad y_{i+1}=-\frac{x_{i+1}}{x_i}\quad(1\le i\le N-2)\,, \qquad y_N=-\frac{z}{x_{N-1}}\,,\\ \tilde{y}_1 &= -\frac{\tilde{x}_1}{x_1}\,, \qquad \tilde{y}_{i+1}=-\frac{\tilde{x}_{i+1}\,x_i}{\tilde{x}_i\,x_{i+1}}\quad(1\le i\le N-2)\,, \quad \tilde{y}_N=-\frac{\tilde{z}\,x_{N-1}}{z\,\tilde{x}_{N-1}}\,, \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} \langle h_i,j\rangle &= \frac{1}{2}\langle h_i,\frac{a}{\epsilon_1}-\rho\rangle\,, \qquad \langle h_i,\tilde{\jmath}\rangle=\frac{1}{2}\langle h_i,\frac{\tilde{a}}{\epsilon_1}-\rho\rangle\,, \qquad k=-N-\frac{\epsilon_2}{\epsilon_1}\,,\\ \frac{\mu_i}{2\epsilon_1} &= \frac{\kappa_2}{N}+\langle h_i,j_1+\frac{\rho}{2}\rangle\,, \quad \frac{\tilde{\mu}_i}{2\epsilon_1}=-\frac{\kappa_4}{N}+\langle h_i,j_5+\frac{\rho}{2}\rangle\,, \quad \frac{m}{2\epsilon_1}=-\frac{\kappa_3}{N}\,. \end{aligned} \tag{4.43}$$

It is also possible to compare terms of the form $y_i^n\,\tilde{y}_l^p$. The new ingredient is the cross-terms

$$\langle\mathbf{n}';j|\mathcal{V}_{\chi_3}(x,z)|\mathbf{m};\tilde{\jmath}\rangle\,. \tag{4.44}$$

To illustrate how the above computations are affected consider first the case when i,l lie in the range $2,\ldots,N-2$. In this case the cross terms are

$$\langle j|(J_0^{i,i-1})^n V_{\chi_3}(x,z)(J_0^{l-1,l})^p|\tilde{\jmath}\rangle\,. \tag{4.45}$$

Now if $i = l$ then $J_0^{i+1,i}$ and $J_0^{l,l+1}$ do not commute which complicates the computation. Similarly, if $i = l+1$ then although $J_0^{l+2,l+1}$ and $J_0^{l,l+1}$ commute, they both act non-trivially on x_l which again affects the calculation. However, apart from these two special cases, it is easy to see that the computation essentially factorises in the sense the coefficient in front of $y_i^n\, \tilde{y}_l^p$ is simply the product of the coefficient in front of y_i^n in the expansion of the above hypergeometric function times the coefficient in front of $\tilde{y}_l^p$ in the expansion of the other hypergeometric function. This is precisely the structure we found in the instanton expression (2.19).

4.3 One-point conformal block on the torus

Our final example is the one-point block on the torus:

$$\sum_{\mathbf{n};\mathbf{A},\mathbf{n}';\mathbf{A}'} z^n \Big(\prod_{i=1}^{N-1} x_i^{\Upsilon_i}\Big) \langle \mathbf{n}, \mathbf{A}; j | V_{\chi_1}(x,z) \mathcal{K}^\dagger(x,z) | \mathbf{n}', \mathbf{A}'; j\rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j)\,, \tag{4.46}$$

where $\langle \mathbf{n}, \mathbf{A}; j| = \langle j | J_{n_1}^{A_1} \cdots J_{n_\ell}^{A_\ell}$ and $n = \sum_i n_i$. The coefficients Υ_i are determined as follows: a generator J_n^{il} in $\langle \mathbf{n}, \mathbf{A}; j|$ contributes $+1$ to Υ_i and -1 to Υ_l, whereas $J_n^{i\pm}$ contributes ± 1 to Υ_i. As for the sl(2) case, we assume that the only effect of the $\mathcal{K}^\dagger$ operator is the introduction of a prefactor, and we therefore focus on the perturbative expansion of the above conformal block without the $\mathcal{K}^\dagger$ insertion. As in previous examples, we start by computing the z-independent terms. The x_1^n terms arise from expressions of the form:

$$\frac{\langle j | (J_0^{1+})^n V_{\chi_1}(x,z) (J_0^{1-})^n | j\rangle}{\langle j | (J_0^{1+})^n (J_0^{1-})^n | j\rangle} = \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(-1)^\ell}{\ell!} \frac{(-2\frac{\kappa_1}{N})_\ell (2\frac{\kappa_1}{N}+1)_\ell}{(-2j^1)_\ell}\,. \tag{4.47}$$

Similarly, the $(x_i/x_{i-1})^2$ terms (for $i = 2, \ldots, N-1$) arise from

$$\frac{\langle j | (J_0^{i,i-1})^n V_{\chi_1}(x,z) (J_0^{i-1,i})^n | j\rangle}{\langle j | (J_0^{i,i-1})^n (J_0^{i-1,i})^n | j\rangle} = \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(-1)^\ell}{\ell!} \frac{(-2\frac{\kappa_1}{N})_\ell (2\frac{\kappa_1}{N}+1)_\ell}{(-2j^i)_\ell}\,. \tag{4.48}$$

One can also compute the terms involving $(z/x_{N-1})^n$:

$$\frac{\langle j | (J_1^{N-1,-})^n V_{\chi_1}(x,z) (J_{-1}^{N-1,+})^n | j\rangle}{\langle j | (J_1^{N-1,-})^n (J_{-1}^{N-1,+})^n | j\rangle} = \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{(-1)^\ell}{\ell!} \frac{(-2\frac{\kappa_1}{N})_\ell (2\frac{\kappa_1}{N}+1)_\ell}{(2[\sum_i^{N-1} j^i] - k)_\ell}\,. \tag{4.49}$$

Using the formula (3.38) it follows that the terms discussed above contribute as

$$(1-x_1)^{2\frac{\kappa_1}{N}}\, {}_2F_1(1 + 2\frac{\kappa_1}{N}, -2j^1 + 2\frac{\kappa_1}{N}; -2j^1; x_1)\,, \tag{4.50}$$

$$(1-\frac{x_{i+1}}{x_i})^{2\frac{\kappa_1}{N}}\, {}_2F_1(1 + 2\frac{\kappa_1}{N}, -2j^{i+1} + 2\frac{\kappa_1}{N}; -2j^{i+1}; \frac{x_{i+1}}{x_i})\,, \tag{4.51}$$

and

$$(1-\frac{z}{x_{N-1}})^{2\frac{\kappa_1}{N}}{}_2F_1(1+2\frac{\kappa_1}{N},2[\sum_i^{N-1}j^i]-k+2\frac{\kappa_1}{N};2[\sum_i^{N-1}j^i]-k;\frac{z}{x_{N-1}})\,. \tag{4.52}$$

By using the identifications (4.29) together with the dictionary

$$\kappa_1=-N\frac{m}{2\epsilon_1},\quad j^i=-\frac{1}{2}+\frac{a_i-a_{i+1}}{2\epsilon_1},\quad k=-N-\frac{\epsilon_2}{\epsilon_1}$$

we find that (up to a prefactor) these expressions are equivalent to the instanton partition function in the $\mathcal{N}=2^*$ SU(N) gauge theory where the corresponding terms take the form (2.14).

5 Asymptotically free SU(N) gauge theories and affine sl(N)

So far we have only discussed conformal $\mathcal{N}=2$ quiver gauge theories. But as we discuss in this section it is also possible to treat non-conformal (asymptotically free) $\mathcal{N}=2$ theories.

The extension of the SU(2) AGT relation to non-conformal theories was carried out in [7]. In this paper Gaiotto conjectured that the instanton partition function for the pure SU(2) theory can be obtained via

$$Z_{\text{inst}}=\langle\Delta,\Lambda|\Delta,\Lambda\rangle\,, \tag{5.1}$$

where the state $|\Delta,\Lambda\rangle$ should satisfy

$$L_1|\Delta,\Lambda\rangle=\Lambda|\Delta,\Lambda\rangle\,,\qquad L_n|\Delta,\Lambda\rangle=0\quad(n\geq 2)\,. \tag{5.2}$$

In an important further development [8] it was shown that the Gaiotto state $|\Delta,\Lambda\rangle$ is a particular state in the Verma module (thereby proving its existence):

$$|\Delta,\Lambda\rangle=\sum_Y\Lambda^nQ_\Delta^{-1}(1^n;Y)|Y,\Delta\rangle\,, \tag{5.3}$$

where Y denotes a partition (Young tableau) $\ell^{n_\ell}\cdots 2^{n_2}1^{n_1}$, where $n=|Y|$ is the number of boxes in Y, $|Y,\Delta\rangle$ denotes the descendant $(L_{-\ell})^{n_\ell}\cdots(L_{-2})^{n_2}(L_{-1})^{n_1}|\Delta\rangle$ of the primary state $|\Delta\rangle$ with conformal dimension Δ, and $Q_\Delta(Y,Y')=\langle Y,\Delta|Y',\Delta\rangle$ is the inner product of descendants (usually called the Gram or Shapovalov matrix) with inverse Q_Δ^{-1}. When combining (5.3) with (5.1) it follows that

$$Z_{\text{inst}}=\sum_{n=0}^\infty Q_\Delta^{-1}(1^n;1^n)\Lambda^{2n}\,. \tag{5.4}$$

(Note that it follows from (5.2) that the only $\langle\Delta,Y|$ that have non-zero inner product with $|\Delta,\Lambda\rangle$ are $\langle\Delta,1^n|$).

The result (5.4) can also be obtained from the AGT relation by sending the masses to infinity in a conformal SU(2) theory [8, 37], and was proven in [38]. The extension to higher rank SU(N) theories was discussed in [39].

The addition of simple surface operators to non-conformal theories is also possible: as in the conformal cases one inserts degenerate states in the dual $2d$ CFT. In [30] it was shown that for the SU(2) theory with a (simple) surface operator, the dual conformal block satisfies a differential equation (the same differential equation was also found earlier in [19], which reflects the fact that for the non-conformal SU(2) gauge theory surface operators obtained by $2d$ and $4d$ defects seems to be associated to the same instanton partition function). Further aspects were studied in the recent paper [17].

Our goal is to generalise the above construction to the non-conformal SU(N) theories with a full surface operator insertion. In other words, we want to find analogues of (5.1)-(5.4) in the module of the affine sl(N) algebra. We should point out that the construction below is in agreement with a result proven in the first paper in [19]. In particular, (5.20) and (5.28) correspond to what is called a Whittaker vector in [19]. However, here we use the language of [7, 8] which is more familiar to physicists. We first study the rank one case.

5.1 Pure SU(2)

As in previous sections, we label the descendants of the primary state $|j\rangle$ by

$$|\mathbf{n},\mathbf{A};j\rangle = J^{A_1}_{-n_1}\cdots J^{A_\ell}_{-n_\ell}|j\rangle\,, \tag{5.5}$$

where we put J^{A}_{-n} to the left of $J^{A'}_{-n'}$ if $n>n'$ or if $A<A'$ and $n=n'$. We also define the matrix (denoted $X_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j)$ in previous sections)

$$Q_j(\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}') = \langle \mathbf{n},\mathbf{A};j|\mathbf{n}',\mathbf{A}';j\rangle\,. \tag{5.6}$$

The following set of descendants will play an important role in what follows

$$|n,p;j\rangle = (J^{+}_{-1})^{p}(J^{-}_{0})^{n}|j\rangle\,. \tag{5.7}$$

We denote the corresponding diagonal component of the inverse of the matrix Q_j, i.e. Q_j^{-1}, by $Q_j^{-1}(n,p;n,p)$.

We propose that the instanton expansion of the pure SU(2) theory in the presence of a (full) surface operator can be obtained from

$$Z_{\rm inst} = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} Q_j^{-1}(n,p;n,p)\,x^n\left(\frac{z}{x}\right)^{p}. \tag{5.8}$$

This expression is the analogue of (5.4) when a full surface operator is present. To test this proposal, we first consider the terms containing only x. Since, $|n,0;j\rangle$ only has a non-zero inner product with its conjugate:

$$Q_j(n,0;n,0) = \langle j|(J^{+}_{0})^{n}(J^{-}_{0})^{n}|j\rangle = n!(-2j)_n(-1)^n\,, \tag{5.9}$$

we find that

$$Q_j^{-1}(n,0;n,0) = [Q_j(n,0;n,0)]^{-1} = \frac{1}{n!(-2j)_n(-1)^n}\,, \tag{5.10}$$

which inserted into our proposal (5.8) leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(-2j)_n} x^n\,. \tag{5.11}$$

Similarly, one can also consider the terms involving only powers of $\frac{z}{x}$. In this case, since $|0,p;j\rangle$ only has non-zero inner product with its conjugate:

$$Q_j(0,p;0,p) = \langle j|(J_1^-)^p(J_{-1}^+)^p|j\rangle = p!(2j-k)_p(-1)^p\,, \tag{5.12}$$

our proposal leads to

$$\sum_{n=0}^{\infty} \frac{(-1)^p}{p!(2j-k)_p}\left(\frac{z}{x}\right)^p\,. \tag{5.13}$$

The above two expressions (5.11) and (5.13) are in perfect agreement with the instanton result (2.13) provided we identify

$$j = \frac{a}{\epsilon_1} - \frac{1}{2}\,, \qquad k = -2 - \frac{\epsilon_2}{\epsilon_1}\,, \qquad x = -\frac{y_1}{(\epsilon_1)^2}\,, \qquad \frac{z}{x} = -\frac{y_2}{(\epsilon_1)^2}\,. \tag{5.14}$$

As a further check we consider all terms of the form $x^n \frac{z}{x}$. In this case there are three states that form a closed subset under the inner product involving $|1,n;j\rangle$, namely

$$|\tilde{1}\rangle = J_{-1}^+(J_0^-)^n|j\rangle, \qquad |\tilde{2}\rangle = J_{-1}^0(J_0^-)^{n-1}|j\rangle, \qquad |\tilde{3}\rangle = J_{-1}^-(J_0^-)^{n-2}|j\rangle. \tag{5.15}$$

The corresponding 3×3 block of Q is

$$\begin{pmatrix} [k-2j+2n]M(n) & M(n) & 0 \\ M(n) & \frac{k}{2}M(n-1) & -M(n-1) \\ 0 & -M(n-1) & [k+2j-2(n-2)]M(n-2) \end{pmatrix} \tag{5.16}$$

where

$$M(n) \equiv \langle j|(J_0^-)^n(J_0^-)^n|j\rangle = (-2j)_n n!(-1)^n. \tag{5.17}$$

From the inverse of (5.16) we find that

$$Q_j^{-1}(n,1;n,1) = -(-1)^n \frac{(4+4j+4k+2jk+k^2-6n-4jn-2kn+2n^2)}{(2j-k)(2+k)(2+2j+k)n!(-2j)_n}\,, \tag{5.18}$$

which leads to

$$-\frac{z}{x}\sum_{n=0}^{\infty} \frac{(-1)^n(4+4j+4k+2jk+k^2-6n-4jn-2kn+2n^2)}{(2j-k)(2+k)(2+2j+k)n!(-2j)_n} x^n\,. \tag{5.19}$$

This result is again in perfect agreement with the instanton result (A.3) provided that we use the identifications (5.14).

As in the case without surface operators it is also possible to construct the analogue of the Gaiotto state (5.2). We denote the corresponding state $|x,z;j\rangle$ and demand that it should satisfy

$$J_0^+|x,z;j\rangle = \sqrt{x}\,|x,z;j\rangle\,, \qquad J_1^-|x,z;j\rangle = \sqrt{\frac{z}{x}}\,|x,z;j\rangle\,, \tag{5.20}$$

where all other J_n^A's that annihilate $|j\rangle$ also annihilate $|x,z;j\rangle$. Finally, the analogue of (5.3) is

$$|x,z;j\rangle = \sum_{\mathbf{n},\mathbf{A}} x^{n/2}\left(\frac{z}{x}\right)^{p/2} Q_j^{-1}(n,p;\mathbf{n},\mathbf{A})|\mathbf{n},\mathbf{A};j\rangle\,, \tag{5.21}$$

which satisfies (5.20).

5.2 Pure SU(N)

The above construction also extends to the pure SU(N) theory. The relevant class of descendants is

$$|\vec{n},p;j\rangle = (J_{-1}^{N-1,+})^p(J_0^{N-2,N-1})^{n_{N-1}}\cdots(J_0^{1,2})^{n_2}(J_0^{1,-})^{n_1}|j\rangle\,. \tag{5.22}$$

We propose that the instanton expansion of the pure SU(N) theory in the presence of a full surface operator can be obtained from

$$Z_{\text{inst}} = \sum_{n_1=0}^{\infty}\cdots\sum_{n_{N-1}=0}^{\infty}\sum_{p=0}^{\infty} Q_j^{-1}(\vec{n},p;\vec{n},p)\,x_1^{n_1}\left(\frac{x_2}{x_1}\right)^{n_2}\cdots\left(\frac{x_{N-1}}{x_{N-2}}\right)^{n_{N-1}}\left(\frac{z}{x_{N-1}}\right)^p. \tag{5.23}$$

Again it is easy to check that the terms involving only one of the N variables match with the instanton results. From an earlier section we know that

$$\begin{aligned}
\langle j|(J_0^{1+})^n(J_0^{1-})^n|j\rangle &= n!(2\langle u_2-u_1,j\rangle)_n(-1)^n\,,\\
\langle j|(J_0^{i,i-1})^n(J_0^{i-1,i})^n|j\rangle &= n!(2\langle u_{i+1}-u_i,j\rangle)_n(-1)^n \quad (i=2\ldots,N-1)\,,\\
\langle j|(J_1^{N-1,-})^n(J_{-1}^{N-1,+})^n|j\rangle &= n!(2\langle u_1-u_N,j\rangle-k)_n(-1)^n\,.
\end{aligned} \tag{5.24}$$

Implementing these results into our proposal (5.23) and using

$$y_1 = -\frac{x_1}{(\epsilon_1)^2}\,, \qquad y_{i+1} = -\frac{1}{(\epsilon_1)^2}\frac{x_{i+1}}{x_i} \quad (1\le i\le N-2)\,, \qquad y_N = -\frac{1}{(\epsilon_1)^2}\frac{z}{x_{N-1}}\,, \tag{5.25}$$

together with the identifications

$$\langle u_i,j\rangle = \frac{1}{2}\langle u_i,\frac{a}{\epsilon_1}-\rho\rangle = \frac{1}{2}(\frac{a_i}{\epsilon_1}-\tfrac{1}{2}[N-2i+1])\,, \qquad k=-N-\frac{\epsilon_2}{\epsilon_1}\,, \tag{5.26}$$

we find

$$\sum_{n=0}^{\infty}\frac{1}{n!\,(a_{i+1}-a_i+\epsilon_1+\epsilon_2\lfloor\frac{i}{N}\rfloor)_n}y_i^n\,, \tag{5.27}$$

which agrees with the instanton result (2.13). Terms of the form $y_i^n y_j$ can also be matched, but we will not give the details here.

Finally, the analogue of the Gaiotto state should satisfy

$$\begin{aligned} &J_0^{1+}|\vec{x},z;j\rangle = \sqrt{x_1}\,|\vec{x},z;j\rangle\,, && J_1^{N-1,-}|\vec{x},z;j\rangle = \sqrt{\frac{z}{x_{N-1}}}\,|\vec{x},z;j\rangle\,, \\ &J_0^{1,2}|\vec{x},z;j\rangle = \sqrt{\frac{x_2}{x_1}}\,|\vec{x},z;j\rangle\,, \quad \cdots && J_0^{N-2,N-1}|\vec{x},z;j\rangle = \sqrt{\frac{x_{N-1}}{x_{N-2}}}\,|\vec{x},z;j\rangle\,, \end{aligned} \tag{5.28}$$

where all other J_n^A's that annihilate $|j\rangle$ also annihilate $|x,z;j\rangle$, and has the expansion

$$|\vec{x},z;j\rangle = \sum_{\mathbf{n},\mathbf{A}} x_1^{n_1/2}\left(\frac{x_2}{x_1}\right)^{n_2/2}\cdots\left(\frac{x_{N-1}}{x_{N-2}}\right)^{n_{N-1}/2}\left(\frac{z}{x_{N-1}}\right)^{p/2} Q_j^{-1}(\vec{n},p;\mathbf{n},\mathbf{A})|\mathbf{n},\mathbf{A};j\rangle\,. \tag{5.29}$$

Let us finally mention that it should be possible to derive the above construction as a limit of a conformal SU(N) theory when the masses are taken to infinity. Note that in the above analysis the operator $\mathcal{K}^\dagger$ played no role, but it may be necessary for more general non-conformal quivers.

6 Summary and outlook

In this paper, building on earlier work [1], we proposed a relation between instanton partition functions in SU(N) quiver gauge theories in the presence of a full surface operator (realised by a $4d$ defect from the $6d$ viewpoint) and (slightly modified) affine sl(N) conformal blocks. Although this proposal passed several highly non-trivial tests it is still conjecture. Possibly one can obtain a proof in special cases, e.g. for the one-point conformal block on the torus along the lines in [40] (extending the result in [20]). Perhaps the most important open problem is to gain a better understanding of the operator $\mathcal{K}^\dagger$.

In the main text we did not specify precisely what the theory is whose conformal blocks reproduce the instanton partition function in the presence of a full surface operator. The reason is that the conformal blocks are completely determined by the symmetry algebra alone. Therefore knowledge of the precise theory was not needed. However, just as in the AGT relation [5], one can speculate that the perturbative piece in the full partition function (involving some extension of [41]) may be related to the three-point parts of the correlation functions. Models with affine sl(2) symmetry include the H_3^+ (or SL$(2,\mathbb{C})$/SU(2)) WZNW model [42], as well as the $SL(2,\mathbb{R})$ WZNW model (see e.g. [43]).

In section 3.4 we checked that at the level of the instanton partition function there is no distinction between surface operators arising from $2d$ and $4d$ defects for theories with gauge group SU(2). However, the two realizations seem to be distinct for quiver gauge theories, already for gauge theories with two SU(2) factors. Nevertheless, it is known that affine sl(2) correlation functions and Liouville correlation functions with degenerate field insertions are related [44]. In this map the number of degenerate field

insertions ($2d$ defect surface operators) is larger than the number expected for the description of a $4d$ defect surface operator, that couple to all the gauge group factors in the quiver. However, there is a modification of the map [45] which requires fewer degenerate field insertions provided one also modifies the affine correlation functions. To obtain the right number of degenerate field insertions ($2d$ surface operators) expected for a potential description of a $4d$ surface operator, one needs to replace one of the primary fields by its spectrally-flowed [43] version with one unit of spectral flow. This could possibly be an alternative to the insertion of the operator $\mathcal{K}$ (in [46] some perturbative computations were performed for the the four-point conformal block where one of the primary field is spectrally flowed by one unit). For the higher rank case it looks more problematic to relate affine conformal blocks to conformal blocks involving degenerate primaries in Toda theories [47].

In this paper we only discussed $4d$ $\mathrm{SU}(N)$ quiver gauge theories, but it should also be possible to study the corresponding $5d$ versions. The $5d$ instanton partition functions should be related to topological string partition functions. An important problem is to understand what a full surface operator arising from $4d$ defects corresponds to in the topological string language (the topological string description of a simple surface operator was discussed in [15, 16]; see also [18]). It would also be nice to find a matrix model description [48].

In the recent developments in $\mathcal{N} = 2$ gauge theories, the set of partitions of N (or equivalently, the number of embeddings of $\mathrm{SU}(2)$ inside $\mathrm{SU}(N)$) appears in many places: in the classification of punctures [4]; in the classification of the corresponding degenerate states in the Toda field theories [49]; and also in the classification of surface operators [12]. As we now recall, there is yet another place where the same classification appears. It is known that one can obtain the (quantum) A_{N-1} Toda field theories from a WZNW model by so called Drinfeld-Sokolov reducton (or hamiltonian reduction), see e.g. [50]. In this reduction, the affine $\mathrm{sl}(N)$ algebra turns into the $\mathcal{W}_N$ algebra. What is perhaps less well known is that when the rank is larger than one there are in general many possible reductions. The various possibilities are classified by the number of embeddings of $\mathrm{SU}(2)$ inside $\mathrm{SU}(N)$ (see e.g. [51]). The reduction that gives the standard Toda theory/$\mathcal{W}_N$ algebra [50] corresponds to the principal embedding. The simplest example not of this form arises for rank two and leads to the Polyakov-Bershadsky algebra $\mathcal{W}_3^{(2)}$ [52]. One may wonder if chiral blocks in these more general $\mathcal{W}$ algebras have a dual gauge theory interpretation.

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A Appendix

A.1 The sl(N) Lie algebras

Here we summarise some standard results for the sl(N) (or A_{N-1}) Lie algebras. The root/weight space of the sl(N) Lie algebra can viewed as a $N-1$–dimensional subspace of $\mathbb{R}^N$. The unit vectors of $\mathbb{R}^N$ will be denoted u_i $(i = 1, \ldots, N)$ and satisfy $\langle u_i, u_j \rangle = \delta_{ij}$. The simple roots are $e_i = u_i - u_{i+1}$ $(i = 1, \ldots, N-1)$ and the positive roots are $e_{ij} = u_i - u_j$ (with $1 \leq i < j \leq N$). The Weyl vector, ρ, is half the sum of the positive roots; hence $\rho = \frac{1}{2}\sum_{i=1}^{N}(N - 2i + 1)u_i$. The fundamental weights, Λ_i, are defined as

$$\Lambda_i = u_1 + \cdots + u_i - \frac{i}{N}\sum_{j=1}^{N} u_j \,, \qquad (i = 1, \ldots, N-1) \tag{A.1}$$

and satisfy $\langle \Lambda_i, e_j \rangle = \delta_{ij}$. Note that $\sum_{i=1}^{N-1}\Lambda_i = \rho$. Finally, the weights of the fundamental representation can be chosen as

$$h_i = u_i - \frac{1}{N}\sum_j u_j = \Lambda_1 - \sum_{j=1}^{i-1} e_j \,, \qquad (i = 1, \ldots, N) \tag{A.2}$$

Note that $h_1 = \Lambda_1$ and $\sum_j h_j = 0$.

A.2 Subleading terms in Z_{inst} for SU(N) with a full surface operator

Using the expressions given in section 2, we find that for the pure SU(N) theory the $y_i^n y_j$ terms in the instanton partition function are given by

$$\begin{aligned}
Z_{\text{inst}}^{(1)i,j} \;=\; & \sum_{n=1}^{\infty} \frac{1}{(\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + 1)_n \, n!} \Bigg\{ \left[\frac{\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + n}{\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor} \right]^{\delta_{j,i+1}} \\
& \times \left[\frac{\frac{a_{j+1}}{\epsilon_1} - \frac{a_j}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + 1}{\frac{a_{j+1}}{\epsilon_1} - \frac{a_j}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor - n + 1} \right]^{\delta_{i,j+1}} \frac{1}{\frac{a_{j+1}}{\epsilon_1} - \frac{a_j}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + 1} \\
& + \; \delta_{i,j+1} \left[\frac{n\,(\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + n)}{(\frac{a_{i+1}}{\epsilon_1} - \frac{a_{i-1}}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j+1}{N}\rfloor + 1)(\frac{a_i}{\epsilon_1} - \frac{a_{i-1}}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + 1)(\frac{a_{i-1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} - \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + n - 1)} \right] \\
& + \; \delta_{j,i+1} \left[\frac{n}{(\frac{a_{i+2}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i+1}{N}\rfloor + 1)(\frac{a_i}{\epsilon_1} - \frac{a_{i+1}}{\epsilon_1} - \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor)} \right] \Bigg\} \left(\frac{y_i}{(\epsilon_1)^2} \right)^n \frac{y_j}{(\epsilon_1)^2} \,.
\end{aligned} \tag{A.3}$$

In the above expression, the Kronecker δ is periodically definied i.e. $\delta_{i,j} = \delta_{i+N,j} = \delta_{i,j+N}$. Similarly, for the conformal SU(N) theory with $N_f = 2N$ we find

$$
\begin{aligned}
Z_{\text{inst}}^{(1)i,j} \;=\; & \sum_{n=1}^{\infty} \frac{(\frac{\mu_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + 1)_n (\frac{\tilde{\mu}_i}{\epsilon_1} - \frac{a_i}{\epsilon_1})_n}{(\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + 1)_n \, n!} \Bigg\{ \left[\frac{\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + n}{\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor}\right]^{\delta_{j,i+1}} \\
& \times \left[\frac{\frac{a_{j+1}}{\epsilon_1} - \frac{a_j}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + 1}{\frac{a_{j+1}}{\epsilon_1} - \frac{a_j}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor - n + 1}\right]^{\delta_{i,j+1}} \frac{(\frac{\mu_{j+1}}{\epsilon_1} - \frac{a_j}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + 1)(\frac{\tilde{\mu}_j}{\epsilon_1} - \frac{a_j}{\epsilon_1})}{\frac{a_{j+1}}{\epsilon_1} - \frac{a_j}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + 1} \\
& + \delta_{i,j+1} \left[\frac{n\,(\frac{a_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + n)}{(\frac{a_{i+1}}{\epsilon_1} - \frac{a_{i-1}}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j+1}{N}\rfloor + 1)(\frac{a_i}{\epsilon_1} - \frac{a_{i-1}}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + 1)(\frac{a_{i-1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} - \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + n - 1)}\right] \\
& \times \left[\frac{n\,(\frac{\mu_{i+1}}{\epsilon_1} - \frac{a_{i-1}}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j+1}{N}\rfloor + 1)(\frac{\mu_i}{\epsilon_1} - \frac{a_{i-1}}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor + 1)(\frac{\tilde{\mu}_{i-1}}{\epsilon_1} - \frac{a_{i-1}}{\epsilon_1})(\frac{\tilde{\mu}_i}{\epsilon_1} - \frac{a_{i-1}}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{j}{N}\rfloor)}{(\frac{\mu_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor + n)(\frac{\tilde{\mu}_i}{\epsilon_1} - \frac{a_i}{\epsilon_1} - n + 1)}\right] \\
& + \; \delta_{j,i+1} \left[\frac{n\,(\frac{\mu_{i+2}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i+1}{N}\rfloor + 1)(\frac{\tilde{\mu}_{i+1}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor)}{(\frac{a_{i+2}}{\epsilon_1} - \frac{a_i}{\epsilon_1} + \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i+1}{N}\rfloor + 1)(\frac{a_i}{\epsilon_1} - \frac{a_{i+1}}{\epsilon_1} - \frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor)}\right]\Bigg\} (-y_i)^n \, (-y_j)\,. \qquad \text{(A.4)}
\end{aligned}
$$

A.3 Computations of affine conformal blocks: technical details

In this appendix we collect some selected details of the computations of affine conformal blocks performed in sections 3 and 4.

A rearrangement formula that we repeatedly used is the Zassenhaus formula

$$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]} e^{\frac{1}{6}(2[Y,[X,Y]]+[X,[X,Y]])} \cdots \qquad \text{(A.5)}$$

In the computations we also repeatedly used manipulations of the type

$$(J_0^+)^p (J_0^-)^n |j\rangle = n\,(2j-n+1)\,(J_0^+)^{p-1}(J_0^-)^{n-1}|j\rangle\,. \qquad \text{(A.6)}$$

As an example, for the x^n terms in the $\widehat{\mathrm{sl}}(2)$ four-point conformal block on the sphere the piece involving $\mathcal{K}$ is computed as follows:

$$
\begin{aligned}
\langle j|(J_0^+)^n e^{-xJ_0^-} V_{j_3}(x,z)|j_4\rangle &= \sum_{p=0}^{n} \frac{(-x)^p}{p!} \langle j|(J_0^+)^n (J_0^-)^p V_{j_3}(x,z)|j_4\rangle \\
= \;\; & \sum_{p=0}^{n} \frac{(-x)^p}{p!} \frac{n!\,(-2j)_n\,(-1)^n}{(n-p)!\,(-2j)_{n-p}\,(-1)^{n-p}} \langle j|(J_0^+)^{n-p} V_{j_3}(x,z)|j_4\rangle \\
= \;\; & n!\,(-2j)_n \sum_{p=0}^{n} \frac{(-x)^p}{p!} \frac{(-1)^p}{(n-p)!(-2j)_{n-p}} (j_4-j_3-j)_{n-p}(-x)^{n-p} \qquad \text{(A.7)} \\
= \;\; & n!\,(-2j)_n\,(-x)^n \sum_{p=0}^{n} \frac{(-1)^p}{p!} \frac{(j_4-j_3-j)_{n-p}}{(n-p)!(-2j)_{n-p}} = x^n\,(-2j-[j_4-j_3-j])_n\,.
\end{aligned}
$$

For the mixed term of the form $z\,x^n$ in the $\widehat{\mathrm{sl}}(2)$ four-point conformal block on the

sphere the terms one needs to compute are:

$$\begin{aligned}
\langle 1|\mathcal{K}(x,z)V_{j_3}(x,z)|j_4\rangle &= z\,x^n\,(k-j_3-j_4-j+n+1)(j_3-j_4-j)_{n+1}\,,\\
\langle 2|\mathcal{K}(x,z)V_{j_3}(x,z)|j_4\rangle &= z\,x^n\,(j+j_4-k/2-n)(j_3-j_4-j)_n\,,\\
\langle 3|\mathcal{K}(x,z)V_{j_3}(x,z)|j_4\rangle &= z\,x^n\,(-j_3+j_4+j-n+1)(j_3-j_4-j)_{n-1}\,,
\end{aligned} \tag{A.8}$$

as well as

$$\begin{aligned}
\langle j_1|V_{j_2}(1,1)|1\rangle &= (j-j_2-j_1-n-1)(j_1-j_2-j)_{n+1}\,,\\
\langle j_1|V_{j_2}(1,1)|2\rangle &= (j-j_1-n)(j_1-j_2-j)_n\,,\\
\langle j_1|V_{j_2}(1,1)|3\rangle &= (j_1-j_2-j+n-1)(j_1-j_2-j)_{n-1}\,.
\end{aligned} \tag{A.9}$$

Finally, the 3×3 Gram matrix $X_{rs}=\langle r|s\rangle$ becomes:

$$\begin{pmatrix} [k-2j+2n+2]M(n+1) & M(n+1) & 0 \\ M(n+1) & \frac{k}{2}M(n) & -M(n) \\ 0 & -M(n) & [k+2j-2n+2)]M(n-1) \end{pmatrix} \tag{A.10}$$

where

$$M(n) \equiv \langle j|(J_0^-)^n(J_0^-)^n|j\rangle = (-2j)_n n!(-1)^n\,. \tag{A.11}$$

The computation of the $x_1^n x_2$ terms in the $\widehat{\mathrm{sl}}(N)$ four-point conformal block is very similar. The relevant descendants are

$$|1\rangle = J_0^{12}(J_0^{1-})^n|j\rangle, \qquad |2\rangle = J_0^{2-}(J_0^{1-})^{n-1}|j\rangle\,, \tag{A.12}$$

and the terms one needs to compute are:

$$\begin{aligned}
\langle j_1|V_{\chi_2}(1,1)\mathcal{K}^\dagger(1,1)|1\rangle &= (-1)^n\left(\tfrac{2\kappa_2}{N}+2\langle h_2,j_1\rangle-2\langle h_1,j\rangle\right)_n\left(\tfrac{2\kappa_2}{N}-2\langle h_2,j\rangle+2\langle h_3,j_1\rangle-n\right)\\
\langle j_1|V_{\chi_2}(1,1)\mathcal{K}^\dagger(1,1)|2\rangle &= (-1)^n\left(\tfrac{2\kappa_2}{N}+2\langle h_2,j_1\rangle-2\langle h_1,j\rangle\right)_n\,,
\end{aligned} \tag{A.13}$$

as well as

$$\begin{aligned}
\langle 1|V_{\chi_3}(x,z)|j_4\rangle &= (-x_1)^{n-1}x_2\left(-\tfrac{2\kappa_3}{N}+2\langle h_1,j_4\rangle-2\langle h_1,j\rangle\right)_n\left(\tfrac{2\kappa_3}{N}-2\langle h_2,j_4\rangle+2\langle h_2,j\rangle+n\right)\\
\langle 2|V_{\chi_3}(x,z)|j_4\rangle &= (-x_1)^n x_2\left(-\tfrac{2\kappa_3}{N}+2\langle h_1,j_4\rangle-2\langle h_1,j\rangle\right)_n\,.
\end{aligned} \tag{A.14}$$

Finally, the 2×2 Gram matrix $X_{rs}=\langle r|s\rangle$ with $r,s=1,2$ becomes:

$$\begin{pmatrix} (2\langle h_2,j\rangle-2\langle h_3,j\rangle+n)\,S(n) & -S(n) \\ -S(n) & (2\langle h_1,j\rangle-2\langle h_3,j\rangle-n+1)\,S(n-1) \end{pmatrix} \tag{A.15}$$

where

$$S(n) \equiv (-1)^n n!\,(-2\langle h_1,j\rangle+2\langle h_2,j\rangle)_n\,. \tag{A.16}$$

A.4 Liouville conformal blocks with degenerate operators

Consider a five-point Liouville conformal block where one of the insertions is a degenerate field, i.e.

$$\langle\alpha_1|V_{\alpha_2}(1)V_{-\frac{b}{2}}(x)V_{\alpha_3}(z)|\alpha_4\rangle\,. \tag{A.17}$$

We insert two complete sets of states, yielding

$$\sum_{\mathbf{n},\mathbf{n}',\mathbf{p},\mathbf{p}'}\langle\alpha_1|V_{\alpha_2}(1)|\mathbf{n};\sigma\rangle X^{-1}_{\mathbf{n};\mathbf{n}'}(\sigma)\langle\mathbf{n}';\sigma|V_{-\frac{b}{2}}(x)|\mathbf{p};\tilde{\sigma}\rangle X^{-1}_{\mathbf{p};\mathbf{p}'}(\tilde{\sigma})\langle\mathbf{p}';\tilde{\sigma}|V_{\alpha_3}(z)|\alpha_4\rangle\,, \tag{A.18}$$

where the sum is over partitions $\mathbf{n}=(n_1,n_2,\ldots)$ with $1\leq n_1\leq n_2\leq\ldots\leq n_r$ and $|\mathbf{n};\sigma\rangle$ are descendants of the primary state $|\sigma\rangle$, i.e. $|\sigma,\mathbf{n}\,\rangle\equiv L_{-n_1}L_{-n_2}\ldots L_{-n_r}|\sigma\rangle$. $X^{-1}_{\mathbf{n};\mathbf{n}'}(\sigma)$ is the inverse of the Gram matrix $X_{\mathbf{n};\mathbf{n}'}(\sigma)=\langle\mathbf{n};\sigma|\mathbf{n}';\sigma\rangle$. The matrix $X_{\mathbf{p};\mathbf{p}'}(\tilde{\sigma})$ and the states $|\mathbf{p};\tilde{\sigma}\rangle$ are defined in a similar way. The terms in (A.18) with $\mathbf{p}=\mathbf{p}'=0$ depend only on x and sum up to $\langle\alpha_1|V_{\alpha_2}(1)V_{-\frac{b}{2}}(x)|\tilde{\sigma}\rangle$. The BPZ [53] equation implies that

$$\langle\alpha_1|V_{\alpha_2}(1)V_{-\frac{b}{2}}(x)|\tilde{\sigma}\rangle=x^{b\tilde{\sigma}}(1-x)^{b\alpha_2}\,G(x) \tag{A.19}$$

where $G(x)$ satisfies the hypergeometric differential equation. The solution defined in a neighbourhood of $x=0$ that we need is

$$G(x)=\,{}_2F_1(A,B;C;x) \tag{A.20}$$

with

$$A=b\,(-\alpha_1+\alpha_2+\tilde{\sigma}-\frac{b}{2})\,,\quad B=b\,(\alpha_1+\alpha_2+\tilde{\sigma}-\mathcal{Q}-\frac{b}{2})\,,\quad C=b\,(2\tilde{\sigma}-b)\,, \tag{A.21}$$

where $\mathcal{Q}=b+\frac{1}{b}$. In order to match this component of the conformal block to the instanton partition function with $y_2=0$, cf. (2.15), we use the relations $\epsilon_1=\frac{1}{b}$, $\epsilon_2=b$ and

$$x=-y_1,\quad \alpha_1=\frac{b}{2}+\frac{\tilde{\mu}_1-\mu_2}{2},\quad \alpha_2=\frac{\mathcal{Q}}{2}+\frac{\tilde{\mu}_1+\mu_2}{2},\quad \tilde{\sigma}=\frac{\mathcal{Q}}{2}-a_1\,. \tag{A.22}$$

The terms in (A.18) with $\mathbf{n}=\mathbf{n}'=0$ are a power series in $\frac{x}{z}$ that sums up to $\langle\sigma|V_{-\frac{b}{2}}(x)V_{\alpha_3}(z)|\alpha_4\rangle$. Imposing the BPZ equation we have

$$\langle\sigma|V_{-\frac{b}{2}}(x)V_{\alpha_3}(z)|\alpha_4\rangle=\left(\frac{x}{z}\right)^{b\alpha_4}\left(1-\frac{x}{z}\right)^{b\alpha_3}\,H\left(\frac{x}{z}\right) \tag{A.23}$$

where $H\left(\frac{x}{z}\right)$ satisfies the hypergeometric differential equation. We consider the solution that is defined around $\frac{x}{z}=\infty$; in details

$$H\left(\frac{x}{z}\right)=\left(\frac{x}{z}\right)^{-C_1}\,{}_2F_1(C_1,C_1+1-D_1;C_1-C_2+1;\frac{z}{x})\,, \tag{A.24}$$

where

$$C_1=b\,(-\sigma+\alpha_3+\alpha_4-\frac{b}{2})\,,\quad C_2=b\,(\sigma+\alpha_3+\alpha_4-\mathcal{Q}-\frac{b}{2})\,,\quad D_1=b\,(2\alpha_4-b)\,. \tag{A.25}$$

Considering the dictionary

$$\frac{z}{x} = -y_2, \quad \alpha_3 = \frac{\mathcal{Q}}{2} + \frac{\mu_1 + \tilde{\mu}_2}{2}, \quad \alpha_4 = \mathcal{Q} + \frac{\mu_1 - \tilde{\mu}_2}{2}, \quad \sigma = \frac{\mathcal{Q}}{2} - a_1 - \frac{b}{2} \quad \text{(A.26)}$$

we reproduce the instanton partition function (2.15) depending on y_2.

We have also computed the $z\,x^m$ terms with the result that the instanton partition function Z is equal to the Liouville block up to a prefactor, i.e.

$$Z \;=\; (1-z)^W (1-\frac{z}{x})^{2b\alpha_3} x^{b\tilde{\sigma}} (1-x)^{b\alpha_2} \langle \alpha_1 | V_{\alpha_2}(1) V_{b/2}(x) V_{\alpha_3}(z) | \alpha_4 \rangle \,, \quad \text{(A.27)}$$

where

$$W = -\alpha_2\alpha_3 + \frac{3}{8}(2\alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4 + b)(-2\alpha_1 - 2\alpha_2 - 2\alpha_3 + 2\alpha_4 + b)\,.$$

This result was obtained by expressing the Virasoro L_{-n_i} operators as differential operators and showing that

$$Z \;=\; (1-z)^W (1-\frac{z}{x})^{2b\alpha_3} (G(x) + z\, T_x^{(1)} + z^2\, T_x^{(2)} + \cdots) \quad \text{(A.28)}$$

where $G(x)$ is the hypergeometric function defined in (A.20) and

$$T_x^{(1)} = \frac{1}{x^{b\tilde{\sigma}}(1-x)^{b\alpha_2}} \nabla_x \left(x^{b\tilde{\sigma}} (1-x)^{b\alpha_2} G(x) \right), \quad \text{(A.29)}$$

with

$$\nabla_x \equiv [(\Delta(\tilde{\sigma}) + \Delta(\alpha_3) - \Delta(\alpha_4)] \frac{1}{\Delta(\tilde{\sigma})} [(1-x)\partial_x + \Delta(\alpha_1) - \Delta(\alpha_2) - \Delta(\tilde{\sigma}) - \Delta(-\frac{b}{2})]\,. \quad \text{(A.30)}$$

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